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Fitting Height of Solvable Groups Admitting Fixed-Point-Free Automorphism Groups*

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INTRODUCTION

This paper is concerned with the Fitting height of a solvable group G admitting a fixed-point-free automorphism group A of relatively prime order. A great deal of work has been done by many people toward establishing the following proposition:

Conjecture. Let G and A be finite solvable groups such that $(|G|, |A|) = 1$, $A \neq 1$, and A acts fixed-point-freely on G. Then the Fitting height of G is less than or equal to the number of prime divisors (counting multiplicities) of $|A|$.

Thompson [13] showed that even without the assumption of solvability for G, if A is of prime order, G is nilpotent; i.e., the Fitting height of G is 1.

Shult [11] showed that if A is a Frobenius group whose kernel and complement are of prime order, then the Fitting height of G is at most 2 provided that either $|G|$ is odd or no Fermat prime divides $|A|$.

Berger [2] has shown that if A is nilpotent and Z_p wr Z_p free for all primes p, the conjecture is also true. His result encompasses those of many others.

The major purpose of this paper is to verify the conjecture, if certain conditions on the divisors of $|G|$ and $|A|$ are satisfied, in the case that A is a Frobenius group with cyclic kernel and complement of prime order. This result is contained in Theorem 1.4 and Corollary 1.5 of Part II.

The method of proof is basically that devised by Shult to handle the case in which the kernel of A is of prime order. In Part I, we prove representation theorems for the semidirect product of a solvable group H by a proper subgroup A_1 of A of order prime to $|H|$. Shult's results apply only if A_1 is cyclic; if not, we use Glauberman's work on characters of groups admitting automorphism groups of relatively prime order. These representation theorems are used in

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Part II to solve the Fitting height problem by induction on $|A|$ and $|G| + |A|$. The principal results of Part I are Theorems 1.2, 2.7, and 3.5. Theorem 1.2 is used to show that the other representation theorems of Part I need only be applicable to proper subgroups of A in the solution of certain Fitting height problems. This fact is useful in proving Theorem 2.3 of Part II, which eliminates the prime condition in Shult's proof mentioned above. Furthermore, it enables us to apply the results of Part I in the case that $|A|$ is the product of three distinct primes, as in Theorem 3.3 of Part II.

Theorem 2.7 of Part I is a special case of Theorem 3.5 of that part, and is used in its proof. Theorem 3.5 is used in the proofs of Theorems 2.3 and 3.3 of Part II.

When a result from Part I is referred to in Part II, its number is preceded by I.

I. REPRESENTATION THEOREMS

1. Nonhomogeneity of Certain Modules

Lemma 1.1. *Let G be a finite group that acts transitively on a set Ω of cardinality n. Suppose A is a π-subgroup of G and n is a π'-number, and one of the following holds:*

(1) *A is a Hall π-subgroup of G and there exists a normal complement H to A in G.*

(2) *For some $x \in \Omega$, the stability group G of x in G (the group of all elements in G fixing x) is such that AG is a group.*

(3) *G is solvable.*

Then A fixes some point of Ω.

We leave the proof of this lemma to the reader.

Theorem 1.2. *Let H, A be finite groups such that $(|H|, |A|) = 1$. Suppose A acts on H, and A acts fixed-point-freely on an Abelian normal subgroup M of H. If K is a splitting field for M, and V is a finite-dimensional, irreducible KHA-module such that V_M is a nontrivial KM-module, then V_H is a nonhomogeneous KH-module.*

Proof. Suppose V satisfies the hypotheses of the theorem, but not the conclusion; i.e., $V \cong eW$ as KH-modules, where e is a positive integer and W is an irreducible KH-module. Since $M \lhd H$, by Clifford's theorem [7, 3.4.1(i)], $W_M = C_1 \oplus \cdots \oplus C_n$, where there exists a positive integer f such that $C_i \cong fW_i$, $i \leqslant i \leqslant n$, with $W_1, \ldots, W_n$ pairwise nonisomorphic irreducible

KM-modules. Furthermore [7, 6.4.1(iii)], the C_i, called the Wedderburn components of W with respect to M, are permuted transitively by H. Thus n is the index of the stability group in H of C_1; so $n \mid |H|$.

Now $V_M \cong (V_H)_M \cong (eW)_M \cong eW_M = e(C_1 \oplus \cdots \oplus C_n) \cong e(fW_1 \oplus \cdots \oplus fW_n) \cong ef(W_1 \oplus \cdots \oplus W_n)$ as KM-modules. Since $M \lhd H$ and M is A-invariant, $M \lhd HA$ and Clifford's theorem can be applied to V with respect to M. By the Krull–Schmidt theorem [9, p. 127], there are precisely n Wedderburn components of V with respect to M, and they are permuted transitively by HA.

Let π be the set of all primes that divide $|A|$. Then A is a Hall π-subgroup of HA since $(|H|, |A|) = 1$, and H is a normal complement to A in HA, while n is a π'-number since $n \mid |H|$. Thus by Lemma 1.1(1) applied to HA acting on the n Wedderburn components of V with respect to M, there exists a Wedderburn component U of V with respect to M that is A-invariant. Since M is A-invariant, MA is a group; so U is a KMA-module. Also $U \cong efW_i$ as KM-modules for some i, where W_i is an irreducible KM-module. Since M is Abelian and K is a splitting field for M, $\dim_K W_i = 1$ [7, 3.2.4] and M acts as scalars on W_i and therefore on $U \cong efW_i$. Thus the action of M on U commutes with that of A on U; so $[M, A]$ acts trivially on U. But M is Abelian and $(|M|, |A|) = 1$; so $M = [M, A] \times C_M(A)$ ([7, 5.2.3] contains proof implicitly). Since $C_M(A) = 1$ by hypothesis, $M = [M, A]$; so M acts trivially on U. But $M \lhd HA$; so $C_V(M)$ is a KHA-module [7, 2.6.2]. Since V_M is not trivial by hypothesis, $C_V(M) \neq V$; so since V is an irreducible KHA-module, $C_V(M) = 0$. This contradicts the assertion that M acts trivially on U, and the theorem is proved.

2. A Representation Theorem for an Extra-special Group Admitting a Frobenius Group

If G is a group and K is a field, denote by $\mathcal{M}(KG)$ the set of equivalence classes of KG-modules. If V is a KG-module, denote by $\{V\}$ the element of $\mathcal{M}(KG)$ containing V.

LEMMA 2.1. *Suppose G is a finite group and K is a splitting field for all subgroups of G, where* char $K \nmid |G|$. *Then there exists a finite field extension L of $\mathbb{Q}$ such that L is a splitting field for all subgroups of G, and a bijection $\phi_H\colon \mathcal{M}(LH) \leftrightarrow \mathcal{M}(KH)$ for each subgroup H of G, such that*

(1) *if $V \in \{V\} \in \mathcal{M}(LH)$ and $W \in \phi_H\{V\}$, then* $\dim_K W = \dim_L V$; *H acts trivially on W if and only if H acts trivially on V; and W is an irreducible KH-module if and only if V is an irreducible LH-module*;

(2) *if $H_1 \leqslant H_2 \leqslant G$ and $V \in \{V\} \in \mathcal{M}(LH_2)$, then if $W \in \phi_{H_2}\{V\}$, $W_{H_1} \in \phi_{H_1}\{V_{H_1}\}$*;

(3) *if* $V_i \in \{V_i\} \in \mathscr{M}(LH)$ *and* $W_i \in \phi_H\{V_i\}$ *for* $1 \leqslant i \leqslant j$, *then* $W_1 \oplus \cdots \oplus W_j \in \phi_H\{V_1 \oplus \cdots V_j\}$.

Proof. See Feit [5, (4.1)–(4.4)].

The following lemma follows directly from Clifford's theorem [7; 3.4.1].

LEMMA 2.2. *Suppose* $G = HA$ *is a finite group where* $H \lhd G$ *and* $H \cap A = 1$, *and suppose* V *is a finite-dimensional, irreducible* KG-*module for some field* K. *Let* W *be a Wedderburn component of* V *with respect to* H. *Then the stability group of* W *in* G *is* HB, *where* B *is a subgroup of* A, *and* W *is an irreducible* KHB-*module. Furthermore, if* A *acts fixed-point-freely on* V, B *acts fixed-point-freely on* W.

LEMMA 2.3. *Let* $G = HA$ *be a finite group with* $H \lhd G$, $H \cap A = 1$, *and* A *Abelian. Suppose* K *is an algebraically closed field with* char $K \nmid |G|$ *and* V *is an irreducible* KG-*module such that* V_H *is a homogeneous* KH-*module; so* $V_H \cong eW$, *where* W *is an irreducible* KH-*module and* e *is a positive integer. Then if* $(\dim_K W, |A|) = 1$, V_H *is an irreducible* KH-*module.*

Note. If $(|H|, |A|) = 1$, since $\dim_K W \mid |H|$ by Lemma 2.1 (1) and [7, 4.2.11], automatically $(\dim_K W, |A|) = 1$.

Proof. Let $\mathscr{S}$ be the representation of G over K corresponding to V; so $\mathscr{S}|_H = e\mathscr{R}$, where $\mathscr{R}$ is the representation of H over K corresponding to W. Now for all $g \in G$, $\mathscr{R}^g$ is equivalent to $\mathscr{R}$ [3, (49.7)]. Since $\deg \mathscr{R} = \dim_K W$, $(\deg \mathscr{R}, |A|) = 1$. Then by a result of Glauberman [6, 1], there exists a representation $\mathscr{R}^*$ of G such that $\mathscr{R}^*(h) = \mathscr{R}(h)$ for all $h \in H$. Then $\mathscr{S} = \mathscr{Y} \otimes \mathscr{Z}$, where $\mathscr{Y}$ and $\mathscr{Z}$ are irreducible representations of G such that $\deg \mathscr{Z} = \deg \mathscr{R}$ and $\mathscr{Y}(h) = 1$ for all $h \in H$ [3, (51.7)]. But then $H \leqslant \ker \mathscr{Y}$; so since $HA/H \cong A$ is Abelian, $\deg \mathscr{Y} = 1$. Thus $\deg \mathscr{S} = (\deg \mathscr{Y})(\deg \mathscr{Z}) = 1$ $(\deg \mathscr{R})$. But $\deg \mathscr{S} = e \cdot \deg \mathscr{R}$, so $e = 1$ and $\mathscr{S}|_H = \mathscr{R}$; i.e., V_H is an irreducible KH-module.

The following lemma can be proved by a direct check.

LEMMA 2.4. *Let* $K \leqslant L$ *be finite fields of characteristic* p *and let* A *be a group such that* $p \nmid |A|$. *If* V *is a finite-dimensional* KA-*module, and* $W = V \otimes_K L$, *then* $\dim_L C_W(A) = \dim_K C_V(A)$. *In particular,* A *acts fixed-point-freely on* W *if and only if* A *acts fixed-point-freely on* V.

DEFINITION. G is a Frobenius group if $G = HA$, where H and A are nontrivial finite groups such that $H \lhd G$ and $C_H(a) = 1$ for all $a \in A^{\#}$. (This latter condition implies $H \cap A = 1$.) H is called the kernel of G and A is called a complement of G.

LEMMA 2.5. *Let $A = BQ$ be a Frobenius group with cyclic kernel B and complement Q of prime order, q, and let $L \leqslant \mathbb{C}$ be a finite field extension of the rationals that is a splitting field for all the subgroups of A. Then:*

(1) *If $d \mid |B|$, then $q \mid d - 1$.*

(2) *There are $(|B| - 1)/q$ irreducible characters of A over L whose kernels do not contain B. Let χ be one such. Then $\chi|_B = \sum_{h \in Q} \xi^h$, where $\xi^h|_B^A = \chi$ for all $h \in Q$, and the ξ^h are distinct nonprincipal irreducible characters of B. Furthermore, $\chi|_Q = \rho_Q$.*

Proof. Let $d \mid |B|$. Since B is cyclic, there exists an element $b \in B$ such that $|\langle b\rangle| = d$. Now $\langle b\rangle$ char $B \lhd BQ$; so $\langle b\rangle \lhd BQ$. Thus Q acts on $\langle b\rangle$. Let $h \in Q^{\#}$. $C_B(h) = 1$ since A is a Frobenius group; so $C_{\langle b\rangle}(h) = 1$ and the elements of $\langle b\rangle^{\#}$ are permuted in orbits of length q by h. Thus $q \mid |\langle b\rangle| - 1 = d - 1$.

Now let ξ be a nonprincipal irreducible character of B over L. Then $\xi|_B^A = \chi$, an irreducible character of A over L [5, (25.4)] of degree $|A:B| \deg \xi = q \cdot 1 = q$ since B is Abelian. Also, any irreducible character of A over L whose kernel does not contain B is of the form $\xi|_B^A$ for some nonprincipal irreducible character ξ of B over L [5, (25.4)]. Furthermore, $\chi|_B = \sum_{h \in Q} \xi^h$ where the ξ^h are q distinct nonprincipal irreducible characters of B over L and $\xi^h|_B^A = \chi$ for all $h \in Q$ [5, (9.10), (25.4)]. Then since there are $|B| - 1$ nonprincipal irreducible characters of B over L, there are $(|B| - 1)/q$ irreducible characters of A over L whose kernels do not contain B. Finally, $\chi|_Q = \rho_Q$ for any such χ [5, (25.4)].

LEMMA 2.6. *Let $A = BQ$ be a Frobenius group with cyclic kernel B and complement Q of prime order, q. Suppose A acts irreducibly and faithfully on an elementary Abelian p-group, P, where $p \nmid |A|$. Then:*

(1) *If $b \in B^{\#}$, $C_p(\langle b\rangle) = 1$.*

(2) $|C_P(Q)| = |P|^{1/q}$.

Proof. Let $b \in B^{\#}$; so $\langle b\rangle \lhd BQ$ as above. Then $C_p(\langle b\rangle)$ is $A = BQ$-invariant [7, 2.6.2]. Then since A acts irreducibly on P, $C_P(\langle b\rangle) = 1$ or P, and since A acts faithfully on P, $C_P(\langle b\rangle) = 1$.

Now let K be a finite extension of $\mathbb{F}_p$ that is a splitting field for all the subgroups of A. Consider P as a faithful, irreducible $\mathbb{F}_pA$-module. Then $\mathscr{R} = P \otimes_{\mathbb{F}_p} K$ is a faithful KA-module. Now $C_P(B) = 1$ by (1) above; so $C_{\mathscr{R}}(B)$ is trivial by Lemma 2.4. By Maschke's theorem [7, 3.3.1], $\mathscr{R} = \mathscr{R}_1 \oplus \cdots \oplus \mathscr{R}_j$, where the $\mathscr{R}_i$ are all irreducible KA-modules and $C_{\mathscr{R}_i}(B)$ is trivial for each i, $1 \leqslant i \leqslant j$. Then by Lemma 2.1(1) and (2) and Lemma 2.5, $(\mathscr{R}_i)_Q$ is the regular representation for each i, $1 \leqslant i \leqslant j$. Thus $\dim_K \mathscr{R}_i = q$ and $\dim_K C_{\mathscr{R}_i}(Q) = 1$, $1 \leqslant i \leqslant j$. Thus $\dim_K \mathscr{R} = jq$ and $\dim_K C_{\mathscr{R}}(Q) = j$. Then by Lemma 2.4, $\dim_{\mathbb{F}_p} C_P(Q) = j$, while $\dim_{\mathbb{F}_p} P = jq$. Thus $|P| = p^{jq}$ and $|C_P(Q)| = p^j = |P|^{1/q}$.

THEOREM 2.7. *Suppose S is an extra-special s-group of order s^{2m+1} and $A = BQ$ is a Frobenius group with cyclic kernel B and complement Q of prime order, q, and the following conditions are satisfied:*

(1) $(|S|, |A|) = 1$.

(2) *A acts on S, centralizing $Z(S)$ and acting faithfully and irreducibly on $S/Z(S)$.*

(3) *There exist a splitting field K for all the subgroups of SA with* char $K \nmid |SA|$ *and a finite-dimensional, faithful, irreducible KSA-module V such that V_S is a homogeneous KS-module and A acts fixed-point-freely on V*

Then q is odd and $|B| = (s^m + 1)/(s^{m/q} + 1)$.

Proof. By Lemma 2.1, we can assume $K \leqslant \mathbb{C}$ and K is a finite extension of $\mathbb{Q}$. Consider the $\mathbb{C}SA$-module $V \otimes_K \mathbb{C}$: it is clearly finite dimensional, irreducible, and faithful. Also, $(V \otimes_K \mathbb{C})_S \cong V_S \otimes_K \mathbb{C} \cong (eW) \otimes_K \mathbb{C} \cong e(W \otimes_K \mathbb{C})$ as $\mathbb{C}S$-modules, where e is a positive integer and W is an irreducible KS-module. Thus $W \otimes_K \mathbb{C}$ is an irreducible $\mathbb{C}S$-module; so $(V \otimes_K \mathbb{C})_S$ is a homogeneous $\mathbb{C}S$-module. Now let η be the character of the representation of SA on V and η' be the character of the representation of SA on $V \otimes_K \mathbb{C}$. As complex-valued class functions, $\eta' = \eta$; so $(\eta' |_A, 1_A) = (\eta |_A, 1_A) = 0$ since A acts fixed-point-freely on V; thus A acts fixed-point-freely on $V \otimes_K \mathbb{C}$. Hence we may assume $K = \mathbb{C}$.

Now for any $b \in B^{\#}$, $C_{S/Z(S)}(\langle b \rangle) = 1$ by Lemma 2.6(1). Then since $(|S|, |\langle b \rangle|) = 1$, $C_S(\langle b \rangle)\, Z(S)/Z(S) = 1$ [7, 6.2.2(iv)]; so since $Z(S) \leqslant C_S(A) \leqslant C_S(\langle b \rangle)$, $C_S(\langle b \rangle)/Z(S) = 1$ so $C_S(\langle b \rangle) = Z(S)$ for any $b \in B^{\#}$.

Since $B \lhd A$, $SB \lhd SA$ and $|SA : SB| = |A : B| = |Q| = q$, a prime. Therefore the number of Wedderburn components of V with respect to SB divides q; so it is either q or 1. Thus either the stability group of a Wedderburn component is precisely SB, or V_{SB} is a homogeneous KSB-module.

Assume the former, and let W be one of the q Wedderburn components of V with respect to SB. Then by Lemma 2.2, W is an irreducible KSB-module, and B acts fixed-point-freely on W. Since V is a faithful KSA-module and V_S is homogeneous, W_S is a faithful, homogeneous KS-module [7, 3.4.2]. By Lemma 2.3, since B is Abelian, W_S is an irreducible KS-module. Since $C_S(\langle b \rangle) = Z(S)$ for all $b \in B^{\#}$, we can apply the results of Glauberman [6], which pertain to characters over $\mathbb{C}$.

Let η be the character of the representation of SB on W; so $\eta |_S$ is faithful and irreducible. Then there exist irreducible characters λ of $C_S(B) = Z(S)$ and θ of B and an integer $\epsilon = \pm 1$ such that $\eta |_B = \epsilon\lambda(1)\theta + |B|^{-1}(\eta(1) - \epsilon\lambda(1))\, \rho_B$ [6, 6b]. Since η is faithful and S is extra-special of order s^{2m+1}, $\eta(1) = s^m$ [4, 31.5], and since $Z(S)$ is Abelian, $\lambda(1) = 1$. Therefore $\eta |_B = \epsilon\theta + |B|^{-1} (s^m - \epsilon)\, \rho_B$. Now B acts fixed-point-freely on W; so 1_B is not a constituent of

$\eta\mid_B$. Then since $s^m - \epsilon > 0$, necessarily $\theta = 1_B$, $\epsilon = -1$, and $|B|^{-1}(s^m - \epsilon) = 1$. This implies $|B| = s^m + 1$, or $s^m = |B| - 1$. But $q \mid |B| - 1$ by Lemma 2.5(1); so $q = s$, contradicting the hypothesis that $(|S|, |A|) = 1$.

Therefore, we may assume that V_{SB} is a homogeneous KSB-module. Then since V is an irreducible $KSBQ$-module and Q is abelian, V_{SB} is an irreducible KSB-module by Lemma 2.3. Since $(V_{SB})_S \cong V_S$ is a homogeneous KS-module, another application of Lemma 2.3 yields that V_S is an irreducible KS-module. This further implies that V_{SQ} is an irreducible KSQ-module. We again are in a position to apply Glauberman's results concerning complex characters.

Let η' be the character of the representation of SA on V, and let $\eta'\mid_S = \zeta$; so that ζ is an irreducible faithful character of S. Now if $g \in S$ and $h \in A$, $\zeta(g^h) = \eta'(g^h) = \eta'(g) = \zeta(g)$; i.e., $\zeta^h = \zeta$ for all $h \in A$. We say A fixes ζ. Then there exists a unique character η of SA, called the canonical extension of ζ to SA, such that $\eta\mid_S = \zeta$ and $(\det \eta)(h) = 1$ for all $h \in A$ [6, 2a)]. Now $\eta'\mid_S = \zeta = \eta\mid_S$; so there exists a unique irreducible character β of SA whose kernel contains S such that $\eta' = \eta\beta$ [6, 2c)]. But $\deg \eta' = \deg \zeta = \deg \eta$; so $\deg \beta = 1$. Thus $SA/\ker \beta$ is cyclic; so $SB \leqslant \ker \beta$.

Now $\eta'\mid_{SB}$ is the character of the representation of SB on V_{SB}; so it is irreducible. Since $\eta' = \eta\beta$ and $SB \leqslant \ker \beta$, while $\deg \beta = 1$, we have $\eta'\mid_{SB} = \eta\mid_{SB}$. Thus if $b \in B$, $(\det \eta'\mid_{SB})(b) = (\det \eta\mid_{SB})(b) = (\det \eta)(b) = 1$; so $\eta'\mid_{SB}$ is the canonical extension of ζ to SB since $(\eta'\mid_{SB})_S = \eta'\mid_S = \zeta$. Then $\eta'\mid_{SB}(b)$ is a rational integer for all $b \in B$ [6, 2b)]. Since $C_S(b) = Z(S)$ for all $b \in B^{\#}$, there exists an irreducible character λ of $C_S(B) = Z(S)$ and an irreducible character θ of B such that $(\eta'\mid_{SB})_B = \epsilon\lambda(1)\theta + |B|^{-1}(\zeta(1) - \epsilon\lambda(1))\,\rho_B$, where $\epsilon = \pm 1$. Since ζ is irreducible and faithful, $\zeta(1) = s^m$, and since $Z(S)$ is Abelian, $\lambda(1) = 1$; so $\eta'\mid_B = \epsilon\theta + |B|^{-1}(s^m - \epsilon)\,\rho_B$. Let $b \in B^{\#}$. Then $\rho_B(b) = 0$ [7, 3.6.6]; so $\eta'\mid_B(b) = \epsilon\theta(b)$. Since $\eta'\mid_{SB}(b) = \eta'\mid_B(b)$ is a rational integer, $\epsilon\theta(b)$ and therefore $\theta(b)$ is a rational integer. Now if $2 \mid |B|$, $q \mid 2 - 1 = 1$ by Lemma 2.5(1); so clearly $|B|$ is odd; therefore $|\langle b\rangle|$ is odd. Thus $\theta(b)^{|\langle b\rangle|} = 1$ where $|\langle b\rangle|$ is an odd integer and $\theta(b)$ is a rational integer; so $\theta(b) = 1$ for all $b \in B^{\#}$. Thus $\theta = 1_B$, and $\eta'\mid_B = \epsilon 1_B + |B|^{-1}(s^m - \epsilon)\,\rho_B$.

Now consider $\eta'\mid_A$. Let $\xi_1 = 1_B, \xi_2, \ldots, \xi_{|B|}$ be the irreducible characters of B, numbered so that the induced characters $\xi_2\mid_B^A = \cdots = \xi_{1+q}\mid_B^A$, $\xi_{2+q}\mid_B^A = \cdots = \xi_{1+2q}\mid_B^A, \ldots, \xi_{2+((|B|-1)/q-1)q}\mid_B^A = \cdots \xi_{|B|}\mid_B^A$. (See Lemma 2.5(2).) Let $\chi^i = \xi_{1+iq}\mid_B^A$, $1 \leqslant i \leqslant (|B| - 1)/q$. Let $\chi_1 = 1_A, \chi_2, \ldots, \chi_q$ be the irreducible characters of A whose kernels contain B; so $\chi_1\mid_Q = 1_Q, \chi_2\mid_Q, \ldots, \chi_q\mid_Q$ are the q irreducible characters of Q. By Lemma 2.5(2), $\chi_1, \chi_2, \ldots, \chi_q, \chi^1, \chi^2, \ldots, \chi_{(|B|-1)/q}$ are all the characters of A. Therefore $\eta'\mid_A = a_1\chi_1 + a_2\chi_2 + \cdots + a_q\chi_q + b_1\chi^1 + b_2\chi^2 + \cdots + b_{(|B|-1)/q}\,\chi^{(|B|-1)/q}$ where the a_i's and b_j's are non-negative integers. Since A acts fixed-point-freely on V, $a_1 = 0$. Then $\eta'\mid_B = (\eta'\mid_A)_B = (a_2 + \cdots + a_q)1_B + b_1(\xi_2 + \cdots + \xi_{1+q}) + \cdots + b_{(|B|-1)/q}(\xi_{2+((|B|-1)/q-1)q} + \cdots + \xi_{|B|})$ by Lemma 2.5(2). But also, $\eta'\mid_B = \epsilon 1_B + |B|^{-1}(s^m - \epsilon)\,\rho_B$. Thus $a_2 + \cdots + a_q = \epsilon + B^{-1}(s^m - \epsilon)$ and $b_1 = \cdots =$

$b_{(|B|-1)/q} = |B|^{-1}(s^m - \epsilon)$. Thus $\eta'|_A = a_2\chi_2 + \cdots + a_q\chi_q + |B|^{-1}(s^m - \epsilon)(\chi^1 + \cdots + \chi^{(|B|-1)/q})$, where $a_2 + \cdots + a_q = \epsilon + |B|^{-1}(s^m - \epsilon)$.

Then $\eta'|_Q = (\eta'|_A)|_Q = a_2\chi_2|_Q + \cdots + a_q\chi_q|_Q + |B|^{-1}(s^m - \epsilon)(|B| - 1/q)\rho_Q$ by Lemma 2.5(2). But we can compute $\eta'|_Q$ another way using the fact that $\eta'|_{SQ}$ is irreducible, and $(\eta'|_{SQ})|_S = \zeta$ is irreducible. If $h \in Q^{\#}$, $\langle h\rangle = Q$ and $C_S(h) = C_S(Q)$. Thus there exist an irreducible character $\tilde{\lambda}$ of $C_S(Q)$, an irreducible character $\tilde{\theta}$ of Q, and an integer $\tilde{\epsilon} = +1$ such that $\eta'|_Q = (\eta'|_{SQ})|_Q = \tilde{\epsilon}\tilde{\lambda}(1)\tilde{\theta} + |Q|^{-1}(\zeta(1) - \tilde{\epsilon}\tilde{\lambda}(1))\rho_Q = \tilde{\epsilon}\tilde{\lambda}(1)\tilde{\theta} + (1/q)(s^m - \tilde{\epsilon}\tilde{\lambda}(1))\rho_Q$ [6, 6(b)].

The Three-Subgroup Lemma [7, 2.2.3] can be used to show that $C_S(Q)$ commutes with $[S, Q]$. A standard argument using this fact establishes that $C_S(Q)$ is an extra-special group of order $s^{2m/q+1}$ and that S is a central product of $C_S(Q)$ and $[S,Q]$. This implies that SQ is a central product of $C_S(Q)$ and $[S,Q]Q$. Thus we can regard $\eta'|_{SQ}$ as a character of $X = C_S(Q) \times ([S,Q]Q)$ [7, 3.7.2]. Then $\eta'|_{SQ} = \gamma_1\gamma_2$, with γ_1 and γ_2 faithful, irreducible characters of $C_S(Q)$ and $[S,Q]Q$, respectively. Thus $\eta'|_Q = \gamma_1(1)(\gamma_2|_Q)$. Since γ_1 is faithful and irreducible and $C_S(Q)$ is extra-special, $\gamma_1(1) = s^{m/q}$. Thus $\eta'|_Q = \tilde{\epsilon}\tilde{\lambda}(1)\tilde{\theta} + (1/q)(s^m - \tilde{\epsilon}\tilde{\lambda}(1))\rho_Q = s^{m/q}(\gamma_2|_Q)$. Therefore, $\tilde{\lambda}(1) \neq 1$; so $\tilde{\lambda}(1) = s^{m/q}$ since $C_S(Q)$ is extra-special.

Thus $\eta'|_Q = \tilde{\epsilon}s^{m/q}\tilde{\theta} + (1/q)(s^m - \tilde{\epsilon}s^{m/q})\rho_Q$. But also $\eta'|_Q = a_2\chi_2|_Q + \cdots + a_q\chi_q|_Q + |B|^{-1}(s^m - \epsilon)((|B| - 1)/q)\rho_Q$. Thus $a_2\chi_2|_Q + \cdots + a_q\chi_q|_Q - \tilde{\epsilon}s^{m/q}\tilde{\theta}$ is a multiple of ρ_Q. Then either $\tilde{\theta} = 1_Q$ and $a_2 = \cdots = a_q = -\tilde{\epsilon}s^{m/q}$, or $\tilde{\theta} = \chi_i|_Q$ for some $i > 1$, $a_i = \tilde{\epsilon}s^{m/q}$, and $a_j = 0$ for all $j \neq i$.

In the former case, necessarily $\tilde{\epsilon} = -1$ and $a_2 = \cdots = a_q = s^{m/q}$. Therefore $\epsilon + |B|^{-1}(s^m - \epsilon) = a_2 + \cdots + a_q = (q-1)s^{m/q}$. Consider this modulo q. $q \mid |B| - 1$ by Lemma 2.5(1); so $|B| \equiv 1 \pmod q$, and $|B|^{-1} \equiv 1 \pmod q$. Thus $\epsilon + |B|^{-1}(s^m - \epsilon) \equiv \epsilon + s^m - \epsilon \pmod q$; so $(q-1)s^{m/q} \equiv s^m \pmod q$. But $s^m = (s^{m/q})^q \equiv s^{m/q} \pmod q$; so $-s^m \equiv s^m \pmod q$, so $q \mid 2s^m$. Since $(s, q) = 1$, $q = 2$. But if $q = 2$, $\epsilon + |B|^{-1}(s^m - \epsilon) = (2-1)s^{m/2} = s^{m/2}$; so $|B| = (s^m - \epsilon)/(s^{m/2} - \epsilon)$. If $\epsilon = -1$, $|B| = (s^m + 1)/(s^{m/2} + 1) = (s^{m/2} - 1) + 2/(s^{m/2} + 1)$, which is not an integer; so $\epsilon = 1$. Then $|B| = (s^m - 1)/(s^{m/2} - 1) = s^{m/2} + 1$. But $q \mid |B| - 1$; so $q \mid s^{m/2}$ and $q = s$, a contradiction. Thus $\tilde{\theta} \neq 1_Q$.

Therefore $\tilde{\theta} = \chi_i|_Q$ for some $i > 1$ and $a_i = \tilde{\epsilon}s^{m/q}$, while $a_j = 0$ for all $j \neq i$. Then $\epsilon + |B|^{-1}(s^m - \epsilon) = a_2 + \cdots + a_q = a_i = \tilde{\epsilon}s^{m/q}$; so $|B| = (s^m - \epsilon)/(s^{m/q} - \epsilon)$. We have seen above that this is impossible if $q = 2$; so q is odd.

Now $S/Z(S)$ is an irreducible faithful $\mathbb{F}_s A$-module, and $s \nmid |A|$. Thus $L = \mathbb{F}_s(1^{1/|A|})$ is a splitting field for A [4, 24.11]. Then $S/Z(S) \otimes_{\mathbb{F}_s} L = U \oplus U^{\alpha_2} \oplus \cdots \oplus U^{\alpha_t}$, where U is an irreducible LA-module with associated character χ, and $1, \alpha_2, \ldots, \alpha_t$ is the Galois group of $\mathbb{F}_s(\chi)$ over $\mathbb{F}_s$ [4, 24.14, 24.10]. B acts fixed-point-freely on $S/Z(S)$; so by Lemma 2.4; B acts fixed-point-freely on U. Thus $B \nleqslant \ker \chi$, and by Lemma 2.1, $\chi|_Q = \rho_Q$; so $\chi(h) = 0$ for $h \in Q^{\#}$ [7, 3.6.6]. Thus $\mathbb{F}_s(\chi) = \mathbb{F}_s(\chi|_B)$; so $\mathbb{F}_s(\chi) \leqslant \mathbb{F}_s(1^{1/|B|})$.

Now $t = |\mathbb{F}_s(\chi) : \mathbb{F}_s|$ [10, I.10, I.38]. Therefore, $2m = \dim_{\mathbb{F}_s}(S/Z(S)) =$

$\dim_L(U \oplus \cdots \oplus U^{\alpha_t}) = qt = q \mid \mathbb{F}_s(\chi) : \mathbb{F}_s \mid$. Thus $\mid \mathbb{F}_s(\chi) : \mathbb{F}_s \mid = 2m/q$. Since $\mathbb{F}_s(\chi) \leqslant \mathbb{F}_s(1^{1/|B|})$, $(2m/q) \mid \mid \mathbb{F}_s(1^{1/|B|}) : \mathbb{F}_s \mid$.

Now suppose $\epsilon = 1$; so $\mid B \mid = (s^m - 1)/(s^{m/q} - 1)$ and $\mid B \mid \mid s^m - 1$. Then $\mathbb{F}_s(1^{1/|B|}) \leqslant \mathbb{F}_{s^m}$; so $\mid \mathbb{F}_s(1^{1/|B|}) : \mathbb{F}_s \mid \mid \mid \mathbb{F}_{s^m} : \mathbb{F}_s \mid = m$. Thus $(2m/q) \mid m$. But $m = (2m/q) \cdot (q/2)$, and $q/2$ is not an integer since q is odd. Thus $\epsilon = -1$ and $\mid B \mid = (s^m + 1)/(s^{m/q} + 1)$.

3. A Representation Theorem for a Solvable Group Admitting a Frobenius Group

THEOREM 3.1 (Shult). *Let A be an Abelian group of operators acting on a solvable group H of order prime to $\mid A \mid$, and suppose that $\mid H \mid$ is not divisible by any prime p such that $p^f + 1$ is a divisor of the exponent of A for some positive integer f. Let V be a faithful KHA-module, where K is a splitting field for all the subgroups of HA and* char $K \nmid \mid A \mid$. *Suppose further that*

(1) *V is a sum of equivalent indecomposable KHA-modules;*

(2) *A acts fixed-point-freely on V;*

(3) *if* char $K = r \neq 0$, $O_r(H) = 1$.

Then there exists a nontrivial normal subgroup of A that centralizes H.

Proof. Reference [12, 4.1].

The following lemma is easily established.

LEMMA 3.2. *Suppose V is a KHA-module, where K is a field, $H \lhd HA$, $(\mid H \mid, \mid A \mid) = 1$, and A acts faithfully on H. Then if V_H is a faithful KH-module, V is a faithful KHA-module.*

LEMMA 3.3. *Let H be a solvable group admitting an Abelian group A such that $(\mid H \mid, \mid A \mid) = 1$. Suppose V is a finite-dimensional KHA-module, where K is a splitting field for all the subgroups of HA and* char $K \nmid \mid A \mid$. *Suppose further that*

(1) *V_H is a faithful, homogeneous KH-module;*

(2) *A acts fixed-point-freely on V;*

(3) *$\mid H \mid$ is not divisible by any prime p such that $p^f + 1$ is a divisor of the exponent of A for some positive integer f.*

Then there exists a nontrivial normal subgroup of A centralizing H.

Proof. This lemma is just a slight modification of Theorem 3.1, and is easily derived from that theorem. We proceed by induction on $\dim_K V$. Consider a counterexample to the lemma minimal with respect to this number. Then V is an irreducible KHA-module. For let $W \neq 0$ be an irreducible KHA-submodule of V. By Clifford's theorem [7, 3.4.1(i)], W_H is completely reducible; so W_H is a homogeneous KH-module since V_H is [7, 3.4.2], and W_H is a faithful KH-module since V_H is a faithful, homogeneous KH-module. Thus W satisfies hypothesis (1) of the lemma. Clearly hypotheses (2) and (3) are satisfied; so by minimality of the counterexample, $W = V$.

Now since we have a counterexample to the lemma, A acts faithfully on H. Then by Lemma 3.2 since V_H is a faithful KH-module, V is a faithful KHA-module. Finally, since V is a faithful, irreducible KHA-module, if char $k = r \neq 0$, $O_r(HA) = 1$ [7, 3.1.3]. But $O_r(H)$ char $H \lhd HA$; so $O_r(H) \lhd HA$ and $O_r(H) \leqslant O_r(HA) = 1$. Thus all the hypotheses of Theorem 3.1 are satisfied, while the conclusion fails to hold. This contradiction establishes Lemma 3.3.

The lemma that follows is an easy consequence of basic properties of Frobenius groups.

LEMMA 3.4. *Let $A = BQ$ be a Frobenius group with cyclic kernel B and complement Q of prime order, q. Then the following hold:*

(1) *If C is a subgroup of A not contained in B then $C = (C \cap B) Q^b$ for some $b \in B$. If $C \cap B \neq 1$, C is a Frobenius group with kernel $C \cap B$ and complement Q^b.*

(2) *Suppose C is a subgroup of A such that $C \cap B \neq 1$. Then if $1 \neq N \lhd C$, $1 \neq N \cap B \lhd A$.*

(3) *If b_1 and b_2 are distinct elements of B, $Q^{b_1} \neq Q^{b_2}$. Then if $x \in A \backslash B$, $x \in Q^b$ for some $b \in B$.*

Now we prove a theorem like Lemma 3.3 but with A a Frobenius group with cyclic kernel and complement of prime order instead of an Abelian group:

THEOREM 3.5. *Let H be a solvable group admitting a Frobenius group $A = BQ$ with cyclic kernel B and complement Q of prime order, q. Suppose $(|H|, |A|) = 1$ and V is a finite-dimensional KHA-module, where K is a splitting field for all the subgroups of HA and* char $K \nmid |A|$. *Suppose further that*

(1) *V_H is a faithful, homogeneous KH-module;*

(2) *A acts fixed-point-freely on V;*

(3) *H has no section that is extra-special of order s^{2n+1}, s a prime, such that $(s^n + 1)/(s^{n/q} + 1)$ is a divisor of $|B|$;*

(4) *H has no section that is an elementary Abelian s-group of order s^m such that $(s^m - 1)/(s^{m/q} - 1)$ is a divisor of $|B|$.*

Then there exists a nontrivial normal subgroup of A centralizing H.

Note. If $q = 2$, then (3) and (4) hold immediately, as shown in the proof of Theorem 2.7.

Proof. We proceed by induction on $\dim_K V + |H|$. Consider a counterexample to the theorem such that this number is as small as possible. Then V is an irreducible KHA-module just as in the proof of Lemma 3.3, for the same reasons. Now suppose M is a maximal A-invariant normal subgroup of H; so H/M is a characteristically simple A-module. Thus since H is solvable, H/M is an elementary Abelian group of order s^m for some prime s, and H/M is an irreducible A-module. Let D be the kernel of the action of A on H/M; so $D \lhd A$.

We know $V_H \simeq eU$ as KH-modules, where e is a positive integer and U is an irreducible KH-module. Since $M \lhd H$, $V_M \simeq (V_H)_M \simeq (eU)_M \simeq eU_M \simeq e(f(U_1 \oplus \cdots \oplus U_k))$, where f is a positive integer and $U_1, ..., U_k$ are pairwise nonisomorphic irreducible KM-modules, and $k \mid |H:M|$. Since M is A-invariant, $M \lhd HA$; so by Clifford's theorem and the Krull–Schmidt theorem, V has k Wedderburn components with respect to M, which are permuted transitively by HA. Since M fixes each of these Wedderburn components, they are permuted transitively by $HA/M \simeq (H/M)A$.

Now $C_{H/M}(D) = H/M$; so $H = MC_H(D)$ since $(|H|, |D|) = 1$ [7, 6.2.6(iv)]. Let $y_1 = 1, y_2, ..., y_{s^m} \in C_H(D)$ be a complete set of coset representatives for the normal subgroup M in H; so $\{My_1 = M, My_2, ..., My_{s^m}\} = H/M$. Set $x_i = My_i$, $1 \leqslant i \leqslant s^m$. If C is a Wedderburn component of V with respect to M, $C^{x_i} = C^{My_i} = C^{y_i}$, $1 \leqslant i \leqslant s^m$.

Since $k \mid |H:M|$, $(k, |A|) = 1$. Thus by Lemma 1.1(1) applied to the action of $(H/M)A$ on the k Wedderburn components of V with respect to M, A fixes one of these Wedderburn components; call it C_1 and the others $C_2, ..., C_k$. Then since $(H/M)A = A(H/M)$ is transitive on $\{C_1, ..., C_k\}$ and A fixes C_1, H/M is transitive on $\{C_1, ..., C_k\}$.

But then the stability group of H/M on C_1 is A-invariant since A fixes C_1. The A-irreducibility of H/M and its transitivity on $\{C_1, ..., C_k\}$ imply $k = 1$ or $k = H/M$.

Suppose $k = 1$; then V_M is a homogeneous KM-module. Since V_H is a faithful KH-module, V_M is a faithful KM-module. Thus hypothesis (1) of the theorem is satisfied with M in place of H. Clearly hypotheses (2), (3), and (4) are also satisfied. Then since $|M| < |H|$, by minimality of the counterexample, there exists a nontrivial normal subgroup E of A centralizing M. Then $E \cap B \neq 1$ since A is a Frobenius group with kernel B [5, (25.3)(iii)]. Clearly $M \leqslant C_H(E \cap B)$. If $b \in E \cap B^{\#}$, $\langle b \rangle \leqslant B$ and $\langle b \rangle \lhd A$. Thus $C_{H/M}(\langle b \rangle) = 1$ or $C_{H/M}(\langle b \rangle) = H/M$ by the A-irreducibility of H/M. If $C_{H/M}(\langle b \rangle) = H/M$, $C_H(\langle b \rangle)M = H$, and since $M \leqslant C_H(E \cap B) \leqslant C_H(\langle b \rangle)$, $\langle b \rangle$ centralizes H, contradicting the assumption that we have a counterexample to the theorem. Therefore $C_{H/M}(\langle b \rangle) = 1$ for all $b \in E \cap B^{\#}$, and $C_{H/M}(E \cap B) = 1$. This implies $C_H(E \cap B)M = M$. Then since $M \leqslant C_H(E \cap B)$, $M = C_H(E \cap B)$.

Now $M \lhd H$; so $H = N_H(M)$. Thus $H = C_H(E \cap B)\, C_H(M) = MC_H(M)$ [1, (II.7)]. Let $C_H(M)_s$ be an A-invariant Sylow s-subgroup of $C_H(M)$

[7, 6.2.2(i)]. Since H/M is an s-subgroup and $H = MC_H(M)$, $H = MC_H(M)_s$. Let S be a subgroup of H minimal with respect to being an A-invariant subgroup of $C_H(M)_s$ such that $H = MS$. Since $S \leqslant C_H(M)_s \leqslant C_H(M)$, S centralizes M; so $S \lhd MS = H$. Also, $Z(S)$ centralizes S and M; so $Z(S) \leqslant Z(MS) = Z(H)$. Since $M < H$ and $H = MS$, $S > 1$; thus $1 < Z(S) \leqslant Z(H)$. Now V_H is a sum of isomorphic faithful, irreducible KH-modules, and K is a splitting field for all the subgroups of H; so $Z(H)$ acts as scalars on each of these modules. Therefore $Z(H)$ acts as scalars on the faithful KH-module V_H, and $Z(H)$ is cyclic. Now V is a faithful KHA-module by Lemma 3.2, and $Z(H)$ acts as scalars on V; so $Z(H) \leqslant Z(HA)$. Thus $1 < Z(S) \leqslant Z(H) \leqslant Z(HA)$. Since $E \cap B \leqslant A \leqslant HA$, $E \cap B$ centralizes $Z(S) \leqslant Z(HA)$. Thus $Z(S) \leqslant C_H(E \cap B) = M$. But then $MZ(S) = M < H = MS$; so $Z(S) < S$; i.e., S is non-Abelian.

Let S_0 be any proper A-invariant subgroup of S. By definition of S, $MS_0 < H$; so by definition of M, $MS_0 = M$ and $S_0 \leqslant M$. Since S centralizes M, S centralizes S_0; so $S_0 \leqslant Z(S)$ and $E \cap B$ centralizes S_0. Now if $b \in E \cap B^{\#}$, b centralizes M, and since b does not centralize $H = MS$, b does not centralize S. Thus S is a special s-group [7, 5.3.7]. Since $Z(S) < S$ and $Z(S) \leqslant Z(H)$, which is cyclic, S is an extra-special s-group. Let $|S| = s^{2n+1}$ [7, 5.2.1].

Since S centralizes M, $SC_H(S) = H$, and since $S \lhd H$, V_S is a homogeneous KS-module [7, 3.4.1(iii)]. Also, V_S is a faithful KS-module, since V_H is a faithful KH-module. Now let R be the kernel of the action of A on S; so A/R acts faithfully on S. Since $E \cap B$ does not centralize S, $E \cap B \nleqslant R$ and $B \nleqslant R$. Therefore $R < B$ and A/R is a Frobenius group with cyclic kernel and complement of order q [5, (25.3)(iii)]. Since $R \lhd A$ and S centralizes R, $C_V(R)$ is a KSA-module and is therefore a KSA/R-module.

Now $R < B \leqslant A$ so R is cyclic and acts faithfully on H. Suppose $C_V(R) = 0$. Then by Lemma 3.3 with R in place of A, there exists a prime p and a positive integer f such that $p^f + 1$ divides the exponent of R and p divides $|H|$. But the exponent of R is the order of R; so $p^f + 1 \mid |R| \mid |B|$. Then by Lemma 2.5(1), $q \mid p^f + 1 - 1 = p^f$; so $q = p$, contradicting the hypothesis that $(|H|, |A|) = 1$. Thus $C_V(R) \neq 0$.

Now since V_S is a faithful, homogeneous KS-module, $s \neq \operatorname{char} K$ [7, 3.1.2]. Thus by Maschke's theorem [7, 3.3.1], $C_V(R)_S$ is completely reducible; therefore it is a faithful, homogeneous KS-module [7, 3.4.2]. Since A acts fixed-point-freely on $C_V(R) \leqslant V$, A/R acts fixed-point-freely on $C_V(R)$. Thus by the minimality of our counterexample, $\dim_K C_V(R) + |S| = \dim_K V + |H|$; so $H = S$ and therefore $R = 1$, and $C_V(R) = V$.

Since all A-invariant proper subgroups of $S = H$ are contained in $Z(S)$, $M \leqslant Z(S)$ and therefore $M = Z(S) \leqslant Z(HA) = Z(SA)$. Hence A centralizes M; so $E = A$ and $C_{H/M}(\langle b \rangle) = 1$ for every $b \in B^{\#}$. Thus A acts faithfully as well as irreducibly on $H/M = S/Z(S)$, while A centralizes $Z(S)$. Thus by Theorem 2.7, q is odd and $|B| = (s^n + 1)/(s^{n/q} + 1)$, contradicting hypothesis (3).

Hence $k = |H/M| = s^m$. Relabel the elements of $H/M^{\#}$ so that $C_1^{x_i} = C_i$ for $2 \leqslant i \leqslant s^m$. $C_1^{x_1} = C_1^1 = C_1$ as before. Also, if $a \in A$, $1 \leqslant i, j \leqslant s^m$ and $x_i^a = x_j$, then $C_i^a = (C_1^{x_i})^a = (C_1^a)^{x_i^a} = C_1^{x_j} = C_j$; so the elements of A permute the C_i's in exactly the way they permute the x_i's.

If $D \neq 1$, consider $C_A(x_i)$ for some $i \neq 1$. $D = C_A(H/M) \leqslant C_A(x_i)$; so $D \cap B \leqslant C_B(x_i)$. Since $D \lhd A$, $D \cap B \neq 1$ [5, (25.3)(iii)]. Now $C_B(x_i) \lhd A$; so $C_{H/M}(C_B(x_i)) = 1$ or H/M. Since $x_i \in C_{H/M}(C_B(x_i))^{\#}$, $C_{H/M}(C_B(x_i)) = H/M$; so $C_B(x_i) \leqslant D$. Thus $C_B(x_i) = D \cap B$. Therefore, by Lemma 3.4(1), either $C_A(x_i) = D \cap B$ and is cyclic, or $C_A(x_i) = (D \cap B) Q^b$ for some $b \in B$ and is a Frobenius group with cyclic kernel and complement of order q. If N is a nontrivial normal subgroup of $C_A(x_i)$, then $N \cap B = N \cap C_B(x_i) = N \cap (D \cap B)$ is a nontrivial normal subgroup of A by Lemma 3.4(2). Now $C_A(x_i)$ is the stability group in A of C_i; so since A acts fixed-point-freely on the orbit of C_i under A, $C_A(x_i)$ acts fixed-point-freely on C_i by the method of proof used in Lemma 2.2.

Thus $MC_A(x_i)$ acts on the homogeneous KM-module C_i, with $C_A(x_i)$ acting fixed-point-freely on C_i. Let L_i be the kernel of the action of M on C_i. Since $M \lhd MC_A(x_i)$, $L_i \lhd MC_A(x_i)$, and $(M/L_i)\, C_A(x_i)$ acts on C_i, with M/L_i acting homogeneously and faithfully on C_i, and $C_A(x_i)$ acting fixed-point-freely on C_i. If $C_A(x_i)$ is a Frobenius group, since $\dim_K C_i < \dim_K V$ and $|M/L_i| < |H|$, by minimality of our counterexample some nontrivial normal subgroup N of $C_A(x_i)$ centralizes M/L_i; so there exists $d \in D \cap B^{\#}$ centralizing M/L_i since $N \cap (D \cap B) \neq 1$. If $C_A(x_i)$ is cyclic; i.e., $C_A(x_i) = D \cap B$, by Lemma 3.3 as above some nontrivial normal subgroup of $C_A(x_i) = D \cap B$ centralizes M/L_i. Thus in either case there exists some $d \in D \cap B^{\#}$ centralizing M/L_i. Thus $[M/L_i, \langle d\rangle] = 1$, so $[M, \langle d \rangle] \leqslant L_i$. Now $L_i^{y_i^{-1}}$ is the kernel of the action of M on $C_i^{y_i^{-1}} = C_i^{x_i^{-1}} = C_1$, and $L_i^{y_i^{-1}y_j}$ is the kernel of the action of M on $C_i^{y_i^{-1}y_j} = C_1^{y_j} = C_1^{x_j} = C_j$ for any j, $1 \leqslant j \leqslant s^m$. Since $y_i^{-1}y_j \in C_H(D)$ for all j, and $M < H$, $[M, \langle d \rangle] = [M, \langle d \rangle]^{y_i^{-1}y_j} \leqslant L_i^{y_i^{-1}y_j}$; so $[M, \langle d \rangle] \leqslant \bigcap_{j=1}^{s^m} L_i^{y_i^{-1}y_j} = 1$, since M is faithful on $V = C_1 \oplus \cdots \oplus C_{s^m}$. Now $H = MC_H(D) = MC_H(\langle d \rangle)$; so $[M, \langle d \rangle] = 1$ implies $[H, \langle d \rangle] = 1$, contradicting the assumption that we have a counterexample to the theorem.

Hence $D = 1$; i.e., A acts faithfully on H/M. Now $C_A(x_i)$ acts fixed-point-freely on $C_i \neq 0$ for any $i > 1$; so $C_A(x_i) \neq 1$. But $C_B(x_i) = D \cap B = 1$; so $C_A(x_i) = Q^b$ for some $b \in B$ by Lemma 3.4(1). Since A acts faithfully on H/M, $|C_{H/M}(Q)| = |H/M|^{1/q} = s^{m/q}$. Thus $|C_{H/M}(Q^b)| = |C_{H/M}(Q)^b| = s^{m/q}$ for all $b \in B$. If $b_1, b_2 \in B$, and $b_1 \neq b_2$, then by Lemma 3.4(3), $Q^{b_1} \neq Q^{b_2}$. Thus $\langle Q^{b_1}, Q^{b_2}\rangle$ is not a q-group; so by Lemma 3.4 (3) there exists $b \in B^{\#} \cap \langle Q^{b_1}, Q^{b_2} \rangle$. Then $C_{H/M}(Q^{b_1}) \cap C_{H/M}(Q^{b_2}) = C_{H/M}(\langle Q^{b_1}, Q^{b_2} \rangle) \leqslant C_{H/M}(\langle b \rangle) = 1$. Thus $|\bigcup_{b \in B} C_{H/M}(Q^b)| = |B|\, (s^{m/q} - 1) + 1$. But since every element x_i of $H/M^{\#}$ is centralized by Q^b for some $b \in B$, $s^m = |H/M| = |B|\,(s^{m/q} - 1) + 1$; so $|B| = (s^m - 1)/(s^{m/q} - 1)$, contradicting hypothesis (4) of the theorem. Thus the theorem is proved.

EXAMPLE 3.6. We show by construction that Theorem 3.5 is false if hypothesis (4) is omitted:

Suppose q is an odd prime, s is a prime different from q, and m is a positive integer that is divisible by q such that $q \nmid (s^m - 1)/(s^{m/q} - 1)$. Set $r = s^{m/q}$ and $n = (s^m - 1)/(s^{m/q} - 1)$; so $n = (r^q - 1)/(r - 1) = r^{q-1} + r^{q-2} + \cdots + r + 1$. Since $n = (r - 1)(r^{q-2} + 2r^{q-3} + \cdots + (q - 2)r + (q - 1)) + q$, $(n, r - 1) \mid (n, q) = 1$; so $(n, r - 1) = 1$. Also, $n \equiv 1 \pmod s$; so $s \nmid n$ and $s \nmid nq$.

Consider the field $\mathbb{F}_{s^m}$. The multiplicative group $\mathbb{F}_{s^m}^*$ is cyclic of order $s^m - 1$. Let α be a generator for this group. Then multiplication by $\beta = \alpha^{r-1}$ is a linear transformation of order $(s^m - 1)/(r - 1) = n$ of $\mathbb{F}_{s^m}$ viewed as a vector space over $\mathbb{F}_s$. Let b denote this linear transformation. Now let τ be the field automorphism of $\mathbb{F}_{s^m}$ of order q such that $x^\tau = x^{s^{m/q}}$ for all $x \in \mathbb{F}_{s^m}$. The fixed points of τ are precisely the elements of the subfield $\mathbb{F}_{s^{m/q}}$ of $\mathbb{F}_{s^m}$. τ can also be considered as a linear transformtion of $\mathbb{F}_{s^m}$ viewed as a vector space over $\mathbb{F}_s$.

If $x \in \mathbb{F}_{s^m}$,

$$x^{b^\tau} = ((x^{\tau^{-1}})^b)^\tau = (\beta x^{\tau^{-1}})^\tau = \beta^{s^{m/q}} x = x^{b^{s^{m/q}}}.$$

Thus $\langle\tau\rangle$ acts on $\langle b\rangle$, and $\langle b\rangle\langle\tau\rangle$ is a group of linear transformations of $\mathbb{F}_{s^m}$ over $\mathbb{F}_s$. Now $(b^i)^\tau = b^{i s^{m/q}}$ for any integer i. If $b^i = b^{i s^{m/q}}$, then $\beta^i = \beta^{i s^{m/q}} = (\beta^i)^{s^{m/q}}$; so $\beta^i \in \mathbb{F}_{s^{m/q}}^*$. But then $|\beta^i| \mid s^{m/q} - 1 = r - 1$, while also $|\beta^i| \mid |\beta| = n$. Since $(n, r - 1) = 1, \beta^i = 1$; so $b^i = 1$. Thus τ acts fixed-point-freely on $\langle b\rangle$, and $\langle b\rangle\langle\tau\rangle$ is a Frobenius group with cyclic kernel of order $n = (s^m - 1)/(s^{m/q} - 1)$ and complement $\langle\tau\rangle$ of prime order, q. Set $B = \langle b\rangle$ and $Q = \langle\tau\rangle$ and $A = BQ$. Then A acts faithfully on the elementary Abelian group $\mathbb{F}_{s^m}^+$, and B acts regularly on $\mathbb{F}_{s^m}^+$. Also, $|A| = nq$; so $(|\mathbb{F}_{s^m}^+|, |A|) = 1$. For consistency of notation, let H_0 be an elementary Abelian group of order s^m whose operation is multiplication such that A acts on H_0 as A acts on $\mathbb{F}_{s^m}^+$; i.e., A is faithful on H_0 and B is regular on H_0.

Let $\{h_1 = 1, h_2, \ldots, h_{s^m}\} = H_0$. Since $H_0A = AH_0$, $\{A, Ah_2, \ldots, Ah_{s^m}\}$ is the set of right cosets of A in H_0A. If $g \in H_0A$ and $h_i \in H_0$, then $Ah_ig = Ah_j$ for a unique $h_j \in H_0$. Define a permutation π_g of $1, 2, \ldots, s^m$ via $(i)\pi_g = j$, where $Ah_ig = Ah_j$. Thus $H_0A \to \sum_{s^m}$ homomorphically by $g \to \pi_g$.

Now let M_1 be a cyclic group such that $(|M_1|, |A|) = 1$. Consider $M_1 \operatorname{wr} H_0A$, the semidirect product of $M = M_1 \times M_1 \times \cdots \times M_1$ (s^m factors) by H_0A, where if $g \in H_0A$, $(m_1, m_2, \ldots, m_{s^m})^g = (m_{(1)\pi_g^{-1}}, m_{(2)\pi_g^{-1}}, \ldots, m_{(s^m)\pi_g^{-1}})$. Thus $(m_1, 1, \ldots, 1)^g = (1, \ldots, m_1, \ldots, 1)$, where the m_1 is in the $(1)\pi_g$th entry, since $1 = ((1)\pi_g)\pi_g^{-1}$. Let $M^i = (1, \ldots, M_1, \ldots, 1)$ where the M_1 is in the ith entry; so $M^i \simeq M_1$, $1 \leqslant i \leqslant s^m$. For $a \in A$, $Aa = A$; so $(1)\pi_a = 1$. Thus $(m, 1, \ldots, 1)^a = (m, 1, \ldots, 1)$ for all $m \in M_1$; so A centralizes M^1. Let $N = M^2 \times M^3 \times \cdots \times M^{s^m}$; then NA centralizes M^1 and $MA = M^1 \times NA$. If $h_i \in H_0$, $A(l)\pi_{h_i} = Ah_i$; so $(1)\pi_{h_i} = i$, and $(M^1)^{h_i} = M^i$, $1 \leqslant i \leqslant s^m$.

Suppose $K \leqslant \mathbb{C}$ is a finite extension of $\mathbb{Q}$ that is a splitting field for all subgroups of $MH_0A = M_1 \operatorname{wr} H_0A$. Let W be a faithful, irreducible KM^1-

module; it will be linear since M^1 is cyclic. Let $NA = NBQ$ act on W so that NB is trivial on W and Q acts nontrivially as scalars on W; i.e., $w^\tau = \lambda w$ for all $w \in W$, where $\lambda \in K$, $\lambda \neq 1$ and $\lambda^q = 1$. Thus $MA = M^1 \times NBQ$ acts irreducibly on W with kernel NB; so W is an irreducible KMA-module. Consider the KMH_0A-module $V = W|_{MA}^{MH_0A}$. Since $H_0A = AH_0$, $MH_0A = MAH_0$ and $V = W \oplus W^{h_2} \oplus \cdots \oplus W^{h_{s^m}}$, and MH_0A acts on V as follows: if $v = w^{h_i}$, $v = w^{h_i}$ for some $w \in W$, and for $x \in MH_0A$, $h_ix = yh_j$ for some $y \in MA$ and some $h_j \in H_0$; then $v^x = (w^{h_i})^x = (w^y)^{h_j} \in W^{h_j}$ [4, p. 45].

Now MH_0 acts faithfully on V. For if $x \in MH_0$ and x acts trivially on V, $(w^{h_i})^x = w^{h_i}$ for all $w \in W$ and all $h_i \in H$. This means $h_ix = y_ih_i$ for some $y_i \in MA$ such that $w^{y_i} = w$ for all $w \in W$. Since $x \in MH_0$ and $h_i \in H_0$, $y_i = h_ixh_i^{-1} \in MH_0 \cap MA = M$. Since M acts on W with kernel N, $y_i \in N$. Thus $x = y_i^{h_i} \in N^{h_i}$; so $x \in \bigcap_{i=1}^{s^m} N^{h_i}$. But $N^{h_i} = M^1 \times \cdots \times M^{i-1} \times M^{i+1} \times \cdots \times M^{s^m}$; so $\bigcap_{i=1}^{s^m} N^{h_i} = 1$ and $x = 1$ and indeed V_{MH_0} is a faithful KMH_0-module.

Also, V_{MH_0} is an irreducible KMH_0-module. For let χ be the character of MH_0A on V; so $\chi|_{MH_0}$ is the character of MH_0 on V_{MH_0}. Let ψ be the character of MA on W, and $\gamma = \psi|_M$, the character of M on W_M. Thus $\chi = \psi|_{MA}^{MH_0A}$; so $\chi|_{MH_0} = \gamma|_M^{MH_0}$ [4, 9.1]. Now $\dim_K W = 1$; so γ is irreducible. Then $(\chi|_M, \gamma)_M \geqslant 1$, since W_M is a KM-submodule of V_M. But $\chi|_M = \gamma + \gamma^{h_2} + \cdots + \gamma^{h_{s^m}}$, where $\ker \gamma^{h_i} = (\ker \gamma)^{h_i} = N^{h_i} \neq N$ if $i \neq 1$. Thus $(\chi|_M, \gamma)_M = 1$. Then $(\chi|_{MH_0}, \gamma|_M^{MH_0})_{MH_0} = 1$ by the Frobenius Reciprocity Theorem [4, 9.4(c)]; so $(\chi|_{MH_0}, \chi|_{MH_0})_{MH_0} = 1$, and $\chi|_{MH_0}$ is irreducible. Thus V_{MH_0} is an irreducible, faithful, KMH_0-module.

Next we show that A acts fixed-point-freely on V. Note that $B \leqslant F_{s^m}^*$, which acts on H_0 and is transitive on H_0^*. Then $H_0F_{s^m}^*Q \geqslant H_0BQ = H_0A$. Now $C = C_{F_{s^m}^*}(Q)$ is of order $s^{m/q} - 1 = r - 1$, $|B| = (s^m - 1)/(r - 1)$, and $B \cap C = 1$; so $F_{s^m}^* = B \times C$.

Let $h_1 \in C_{H_0}(Q)^*$ and $h \in H_0^*$. Then there exists $bc \in B \times C$ such that $h = h_1^{bc}$. Thus $1 = [h_1, Q] = [h_1, Q]^{bc} = [h, Q^{bc}] = [h, Q^{cb}] = [h, Q^b]$. Therefore, $Q^b = Q^{bh} < A^h$, and $Q^b \leqslant A \cap A^h$. Now $B^{h^{-1}} \cap B = 1$ since B is regular on H_0; so $B^{h^{-1}} \cap A = 1$ since $B = O_{q'}(A)$. Thus $B \cap A^h = 1$, and $A \cap A^h = Q^b$. Since $AH_0 \cap M = 1$, this implies $A \cap (MA)^h = A \cap MA^h = Q^b$.

Now consider $\chi|_A$. Since $\chi = \psi|_{MA}^{MH_0A}$, by the Mackey Subgroup Theorem [3, (44.2)], $\chi|_A = \sum_{MAhA} (\psi^h|_{(MA)^h \cap A}^A) = \sum_{MAhA} \psi^h|_{Q^b}^A$, the sum over (MA, A) double cosets in MAH_0, which are represented by a subset of H_0. Now Q^b acts as multiplication by $\lambda \neq 1$ on W and also W^h since $[Q^b, h] = 1$. Thus $(\psi^h|_{Q^b}, 1_{Q_b})_{Q_b} = 0$; so by the Frobenius Reciprocity Theorem $(\psi^h|_{Q^b}^A, 1_A)_A = 0$. Therefore, $(\chi|_A, 1_A)_A = 0$; i.e., A acts fixed-point-freely on V.

Now set $H = MH_0$. M and H_0 are both Abelian and $M \lhd MH_0$; so H is solvable, and since $M \lhd MH_0A$ and H_0 admits A, $H = MH_0$ admits A. Since A acts faithfully on H_0, A acts faithfully on H. Since $(|H_0|, |A|) = 1$ and $(|M|, |A|) = (|M_1|, |A|) = 1$, $(|H|, |A|) = 1$. Now K is a splitting field for all the subgroups of $MH_0A = HA$, and we have shown:

(1) V_H is a faithful, homogeneous (in fact irreducible) KH-module;

(2) A acts fixed-point-freely on V.

Also, by picking M_1 so that $(|M_1|, |H_0|) = 1$, we can force all the Sylow subgroups of $MH_0 = H$ to be Abelian; so that H has no extra-special section of any kind. Thus all the hypotheses of Theorem 3.5 hold except (4), while the conclusion fails to hold since A is faithful on H.

Now if $q \nmid s^{m/q} - 1$, since $s^m - 1 \equiv s^{m/q} - 1 \pmod q$, $n = (s^m - 1)/(s^{m/q} - 1) \equiv 1 \pmod q$; so $q \nmid n$. Thus given q and $s \not\equiv 1 \pmod q$, a counterexample to Theorem 3.5 minus hypothesis (4) can be constructed by picking an integer t such that $q \nmid s^t - 1$, and, setting $m/q = t$, using the construction given above.

Finally, given any prime p distinct from s such that $p \nmid |A|$, we can choose M_1 so that $p \nmid |M_1|$ and therefore $p \nmid |MH_0A| = |HA|$, and by Lemma 2.1, there exists a counterexample to Theorem 3.5 minus hypothesis (4) with char $K = p$.

II. FITTING HEIGHT

1. Fixed-Point-Free Action of a Frobenius Group with Cyclic Kernel and Complement of Prime Order

Definition. Let G be a finite group. The subgroup of G generated by all its nilpotent normal subgroups is a nilpotent normal subgroup of G called the Fitting subgroup of G, denoted $F(G)$ [7, 6.1.2]. If G is also solvable, then $F(G) \neq 1$ if $G \neq 1$, since any minimal normal subgroup of G is elementary Abelian and therefore nilpotent. For G solvable, define $F_1(G) = F(G)$, and define $F_{i+1}(G)$ for each $i \geqslant 1$ to be the subgroup of G such that $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$. Thus $1 < F_1(G) < \cdots < F_j(G) = G$ for some integer j, since G is finite. Denote by $f(G)$ the smallest integer j such that $F_j(G) = G$; $f(G)$ is called the Fitting height of G.

The two lemmas that follow are easy to prove.

Lemma 1.1. *Let G be a finite solvable group.*

(1) *If $f(G) > 1$; i.e., G is not nilpotent, then $f(G/F(G)) = f(G) - 1$.*

(2) *If $H \lhd G$, then $f(G/H) \leqslant f(G)$.*

(3) *If $H \leqslant G$, then $f(H) \leqslant f(G)$.*

Lemma 1.2. *Let G_1 and G_2 be finite solvable groups. Then $F(G_1 \times G_2) = F(G_1) \times F(G_2)$, and $f(G_1 \times G_2) = \max(f(G_1), f(G_2))$.*

LEMMA 1.3. *Suppose G is a finite group. Then $\Phi(F(G)) \leqslant \Phi(G)$.*

Proof. Let M be a maximal subgroup of G, and set $L = \bigcap_{x \in G} M^x$; so $L \lhd G$. Then $F(G)L/L \leqslant F(G/L)$. If $F(G/L) = 1$, $F(G) \leqslant L$; so $\Phi(F(G)) \leqslant M$. If $F(G/L) \neq 1$, it is an elementary Abelian p-group for some prime p [7, 6.1.4(ii)]. Therefore $F(G)/L \cap F(G) \simeq F(G)L/L$ is an elementary Abelian p-group; so $\Phi(F(G)) \leqslant L \cap F(G)$ and $\Phi(F(G)) \leqslant M$. Thus $\Phi(F(G))$ is contained in any maximal subgroup of G, and therefore $\Phi(F(G)) \leqslant \Phi(G)$, the intersection of all maximal subgroups of G.

DEFINITION. Let A be a nontrivial finite group; so for some $j \geqslant 1$, $| A | = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ where $p_1, p_2, \ldots, p_j$ are distinct primes. Then let $l(A) = k_1 + k_2 + \cdots + k_j$.

THEOREM 1.4. *Let $A = BQ$ be a Frobenius group with cyclic kernel B and complement Q of prime order, q. Suppose G is a finite solvable group such that $(| G |, | A |) = 1$ and A acts fixed-point-freely on G. Suppose further that:*

(1) *if q is a Fermat prime, of the form $2^f + 1, f > 0$, then G has no extra-special section of order 2^{2f+1};*

(2) *G has no extra-special section of order s^{2n+1}, s a prime, such that $(s^n + 1)/(s^{n/q} + 1)$ is a proper divisor of $| B |$;*

(3) *G has no elementary Abelian section of order s^m, s a prime, such that $(s^m - 1)/(s^{m/q} - 1)$ is a proper divisor of $| B |$.*

Then $f(G) < l(A)$.

COROLLARY 1.5. *Let A be a dihedral group with cyclic maximal subgroup B of odd order. Let G be a finite solvable group such that $(| G |, | A |) = 1$ and A acts fixed-point-freely on G. Then $f(G) \leqslant l(A)$.*

Proof of Corollary 1.5. $A = BQ$, where $| Q | = q = 2$ and Q acts fixed-point-freely on B; so A is a Frobenius group and Theorem 1.4 applies. q is not a Fermat prime; so hypothesis (1) holds vacuously. $(s^n + 1)/(s^{n/2} + 1) = s^{n/2} - 1 + 2/(s^{n/2} + 1)$ is not an integer when s is prime and $n > 0$; so hypothesis (2) holds. Finally, if s is a prime and $s \mid | G |$, $s \nmid | A | = 2 | B |$; so s is odd. Then $(s^m - 1)/(s^{m/2} - 1) = s^{m/2} + 1$ is even; so it is not a divisor of the odd number $| B |$ and hypothesis (3) holds. Therefore $f(G) \leqslant l(A)$ by Theorem 1.4.

Proof of Theorem 1.4. We reduce the theorem to a problem in representation theory, which is solved by means of the results of Part I.

Suppose Theorem 1.4 is false, and G and A are groups satisfying the hypotheses but not the conclusion of the theorem such that $| A |$ and $| G | + | A |$ are minimal with respect to this property. We show that A acts faithfully on G, and that $F(G)$ is an elementary Abelian r-group and $F(G) = F(GA)$. Let A_0 be the kernel of the action of A on G; so A/A_0 acts fixed-point-freely on G. Thus $A_0 \lhd A$ and $A_0 < A$ since $G \neq 1$. Then if $A_0 = B$, $A/A_0 \cong Q$; so G is nilpotent by Thompson's theorem [7, 10.2.1], and $f(G) = 1 \leqslant l(A)$. If $A_0 \neq B$, A/A_0 is a Frobenius group with cyclic kernel and complement of order q [5, (25.3)(iii)]. Then if $A_0 > 1$, $|A/A_0| < |A|$; so by minimality of $|A|$, $f(G) \leqslant l(A/A_0) < l(A)$. Thus $A_0 = 1$; i.e., A acts faithfully on G. Set $\mathcal{N} = |A|$.

Suppose G contains two distinct subgroups, M and N, each of which is minimal with respect to being a nontrivial normal A-invariant subgroup of G. Then A acts fixed-point-freely on G/M and on G/N [7, 6.2.2(iv)]. Since $|G/M| + |A| < |G| + |A|$ and $|G/N| + |A| < |G| + |A|$, $f(G/M) \leqslant l(A)$ and $f(G/N) \leqslant l(A)$. But $g \mapsto (Mg, Ng)$ gives a homomorphism from G into $G/M \times G/N$, whose kernel is $M \cap N = 1$. Thus G is isomorphic to a subgroup of $G/M \times G/N$; so by Lemma 1.1(3), $f(G) \leqslant f(G/M \times G/N)$. But by Lemma 1.2, $f(G/M \times G/N) = \max(f(G/M), f(G/N)) \leqslant l(A)$; so $f(G) \leqslant l(A)$, a contradiction. Thus G contains only one subgroup that is minimal with respect to being a normal, A-invariant subgroup of G; call this subgroup M. M is characteristically simple; so since G is solvable, M is an elementary Abelian r-group for some prime r. Also, any nontrivial normal A-invariant subgroup of G contains a minimal such subgroup, which is necessarily M. Thus if s is a prime distinct from r and $O_s(G) \neq 1$, then $M \leqslant O_s(G)$, which is impossible; so $O_s(G) = 1$. Thus $F(G) = O_r(G)$, an r-group.

Now $\Phi(G)$ char G; so A acts fixed-point-freely on $G/\Phi(G)$. But $F(G/\Phi(G)) = F(G)/\Phi(G)$ [7, 6.1.6(ii)]; so $(G/\Phi(G))/F(G/\Phi(G)) = (G/\Phi(G))/(F(G)/\Phi(G)) \cong G/F(G)$. Thus $f(G/\Phi(G)) - 1 = f(G) - 1$ by Lemma 1.1(1); so $f(G/\Phi(G)) = f(G)$. Therefore $|G/\Phi(G)| + |A| = |G| + |A|$; so $\Phi(G) = 1$. Then $\Phi(F(G)) = 1$ by Lemma 1.3; so since $F(G) = O_r(G)$ is an r-group, $F(G)$ is an elementary Abelian r-group.

Now $G \cap F(GA)$ is nilpotent and normal in G; so $G \cap F(GA) \leqslant F(G)$. But $F(G)$ char $G \lhd GA$; so $F(G) \lhd GA$; hence $F(G) \leqslant G \cap F(GA)$, and $F(G) = G \cap F(GA)$. Let s be a prime distinct from r, and suppose $O_s(GA) \neq 1$. If $s \mid |G|$, $O_s(GA)$ is contained in a Sylow s-subgroup of G, since $(|G|, |A|) = 1$ implies such a subgroup is a Sylow s-subgroup of GA. Thus $O_s(GA) \leqslant G$; so $O_s(GA) \leqslant O_s(G) = 1$, a contradiction. Thus $s \nmid |G|$; so $s \mid |A|$. Therefore $O_s(GA) \leqslant A$ by the same argument. But then since $G \lhd GA$, $[G, O_s(GA)] \leqslant G \cap O_s(GA) = 1$, contradicting the faithfulness of A on G. Thus $F(GA) = O_r(GA)$, and $F(GA) = O_r(GA) \leqslant O_r(G) = F(G)$ by the same argument again. Since $F(G) \leqslant F(GA)$, $F(G) = F(GA)$.

Since GA is solvable, $C_{GA}(F(GA)) \leqslant F(GA)$ [7, 6.1.3], and since $F(GA)$ is Abelian, $C_{GA}(F(GA)) = F(GA)$. Thus $GA/F(GA) = GA/F(G) \cong (G/F(G))A$ acts faithfully on $F(G) = F(GA)$. The elementary Abelian r-group $F(G)$ can therefore be considered as a faithful $\mathbb{F}_r(G/F(G))A$-module. Also, A acts fixed-point-freely on both $F(G)$ and $G/F(G)$. And since $F(G) = F(GA) = O_r(GA)$, $O_r((G/F(G))A) \simeq O_r(GA/F(G)) = 1$.

Finally, we can show that $F(G) = M$, the unique subgroup of G minimal with respect to being normal and A-invariant. Consider $X/M = F(G/M)$, a nilpotent group. Then $O_r(X/M)$ char X/M char G/M; so $O_r(X/M) = R/M$ for some r-group $R \lhd G$. Thus the r-group $F(G)/M$ contains the Sylow r-subgroup R/M of $F(G/M)$. But $F(G)/M \leqslant F(G/M)$; so $R = F(G)$. By minimality of $|G| + |A|$, $f(G/M) < f(G)$; so $f((G/M)/F(G/M)) < f(G/F(G))$ by Lemma 1.1(1). Since $(G/M)/(F(G)/M) \simeq G/F(G)$, $F(G)/M < F(G/M)$, and $F(G) < X$.

Now $F(G)/M$ is the Sylow r-subgroup of X/M; so $F(G)$ is the Sylow r-subgroup of X. Thus the elementary Abelian group $F(G)$ is an $F_r(X/F(G))$-module, where $X/F(G)$ is an r'-group. Since $F(G)$ centralizes M and $M \lhd X$, M is $X/F(G)$-invariant, and $F(G) = M \times N$, where N is $X/F(G)$-invariant [7, 3.3.2]. Thus $N \lhd X$. The nilpotent group X/M has an Abelian Sylow r-subgroup $F(G)/M$; so $F(G)/M \leqslant Z(X/M)$. Therefore, $[X, N] \leqslant [X, F(G)] \leqslant M$, while $[X, N] \leqslant N$; so $N \leqslant Z(X)$.

Since X/M char G/M and M is A-invariant, X is an A-invariant normal subgroup of G. Therefore, $Z(X)$ is also an A-invariant normal subgroup of G. Then if $Z(X) > 1$, $Z(X) \geqslant M$. But then $Z(X) \geqslant M \times N = F(G)$, implying $X/F(G)$ centralizes $F(G)$, contradicting the faithfulness of $G/F(G)$ on $F(G)$. Thus $Z(X) = 1$; so $N = 1$ and $F(G) = M$ as claimed. This means that, as an $F_r(G/F(G))A$-module, $F(G)$ is irreducible as well as faithful. Finally, this implies that $O_r(G/F(G)) = 1$ [7, 3.1.3].

Thus $G/F(G)$, A, $F(G)$, $\mathbb{F}_r$ constitute a counterexample to the following theorem, with $G/F(G)$ in the place of H, $F(G)$ in the place of F, and $\mathbb{F}_r$ in the place of K:

THEOREM 1.6. *Let $A = BQ$ be a Frobenius group with cyclic kernel B and complement Q of prime order, q. Let H be a finite solvable group such that $(|H|, |A|) = 1$ and A acts fixed-point-freely on H. Suppose F is a finite-dimensional, irreducible, fatihful KHA-module, where K is a finite field of characteristic r such that $r \nmid |A|$ and $O_r(H) = 1$. Suppose further that A acts fixed-point-freely on F and*

(1) *if q is a Fermat prime, of the form $2^f + 1, f > 0$, then H has no extra-special section of order 2^{2f+1};*

(2) *H has no extra-special section of order s^{2n+1}, s a prime, such that $(s^n + 1)/(s^{n/q} + 1)$ is a proper divisor of $|B|$;*

(3) *H has no elementary Abelian section of order s^m, s a prime, such that $(s^m - 1)/(s^{m/q} - 1)$ is a proper divisor of $|B|$.*

Then $f(H) \leqslant l(A) - 1$.

Note. Once Theorem 1.6 is established, the hypothesis that F is an irreducible KHA-module can be dropped. For if $O_r(H) = 1$, F is the Sylow r-subgroup of $F(FH)$. Since H is faithful on F, $F(FH)$ is an r-group; so $F(FH) = F$. Therefore, $f(H) = f(FH) - 1 \leqslant l(A) - 1$ by Lemma 1.1(1) and Theorem 1.4, which follows from Theorem 1.6. The extra hypothesis is originally included in Theorem 1.6 because it considerably simplifies the induction proof. My thanks to the referee for pointing out the efficiency of this approach.

Proof. We have shown that if Theorem 1.4 is false, so is Theorem 1.6; thus if Theorem 1.6 is true, so is Theorem 1.4. Now suppose H, A, F, K constitute a counterexample to Theorem 1.6 with $|A|$ and $|A| + \dim_K F$ as small as possible. Thus $|A| \leqslant \mathscr{N}$. Let L be a finite field of characteristic r such that L is a splitting field for all subgroups of HA [3, (69.11)]. Then $F \otimes_K L$ is a finite-dimensional, faithful LHA-module, and A acts fixed-point-freely on $F \otimes_K L$ by Lemma I.2.4. Also, $\dim_L F \otimes_K L = \dim_K F$.

Now $F \otimes_K L = F_1 \oplus \cdots \oplus F_K$, where the F_i are algebraically conjugate, absolutely irreducible, LHA-modules [3, (70.15)]. Thus each F_i is a faithful LHA-module. But then H, A, F_i, L satisfy the hypotheses of the theorem; so $F_i = F \otimes_K L$ by minimality of $|A| + \dim_K(F)$. Thus we may assume that K is a splitting field for all subgroups of HA.

Now $F = V_1 \oplus \cdots \oplus V_e$, where the V_i, $1 \leqslant i \leqslant e$, are the Wedderburn components of F with respect to H. Let T_i be the stability group in HA of V_i, $1 \leqslant i \leqslant e$. Then $H \leqslant T_i \leqslant HA$; so $T_i = H(T_i \cap A)$. Let $a \in A$. Then $T_i{}^a = H(T_i \cap A)^a$ is the stability group in HA of the Wedderburn component $V_i{}^a = V_j$, some j, $1 \leqslant j \leqslant e$. Thus there exists a Wedderburn component, which we denote V_1, such that if $q \mid |T_i \cap A|$ for all i, $1 \leqslant i \leqslant e$, then $Q \leqslant T_1 \cap A$. Now any Wedderburn component is equal to $V_1{}^a$ for some $a \in A$; so $|T_i \cap A| = |T_1 \cap A|$ for all i, $1 \leqslant i \leqslant e$.

Since A acts transitively on $\{V_1, \dots, V_e\}$, $|A : T_1 \cap A| = e$. Since A acts fixed-point-freely on V, $T_1 \cap A$ acts fixed-point-freely on V_1 by Lemma I.2.2. Let $A_1 = T_1 \cap A$. Since V_1 is nonzero, $A_1 \neq 1$. Thus there are four possibilities:

(a) $A_1 = BQ = A$;

(b) $A_1 \leqslant B$;

(c) $A_1 = Q$;

(d) $A_1 = B_1 Q$, where $1 < B_1 < B$.

Now let $a_1 = 1, a_2, \dots, a_e$ be a right transversal for A_1 in A (so it is a right transversal for $HA_1 = T_1$ in HA) such that $V_1^{a_i} = V_i$, $1 \leqslant i \leqslant e$. Let δ_1 be the representation of HA_1 on V_1. Therefore $H \cap \ker \delta_1$ is the kernel of the action of H on V_1. Thus $(H \cap \ker \delta_1)^{a_i}$ is the kernel of the action of H on V_i, $1 \leqslant i \leqslant e$; so $\bigcap_{i=1}^{e} (H \cap \ker \delta_1)^{a_i} = 1$ since H acts faithfully on $F = V_1 \oplus \cdots \oplus V_e$.

Now $H \lhd HA_1$ and $\ker \delta_1 \lhd HA_1$; so $H \cap \ker \delta_1 \lhd HA_1$ and A_1 acts on $H/H \cap \ker \delta_1$. Thus $(H/H \cap \ker \delta_1) A_1$ acts on V_1. Since V_1 is a Wedderburn component of F with respect to H, $(V_1)_H$ is a homogeneous KH-module; so $(V_1)_{H/H \cap \ker \delta_1}$ is a faithful, homogeneous $K(H/H \cap \ker \delta_1)$-module.

If (a) holds, $A_1 = A$ and $e = | A : A_1 | = 1$; so $V_1 = F$. Thus F is a finite-dimensional, faithful, irreducible KHA-module such that F_H is homogeneous. Let M be a minimal normal A-invariant subgroup of H. Since H is solvable, M is elementary Abelian. Since A acts fixed-point-freely on H, A acts fixed-point-freely on M. Since F is a faithful KHA-module, F_M is nontrivial. Then by Theorem I.1.2, F_H is not a homogeneous KH-module. Thus case (a) cannot occur.

If (b) holds, A_1 is a cyclic group, contained in B. If p is a prime such that for some positive integer f, $p^f + 1$ divides the exponent of A_1 (which is $| A_1 |$), then $q \mid p^f + 1 - 1 = p^f$; so $q = p$. Thus $p \nmid | H/H \cap \ker \delta_1 |$. Thus by Lemma I.3.3 applied to $H/H \cap \ker \delta_1$, A_1, V_1, K in place of H, A, V, K, A_1 does not act faithfully on $H/H \cap \ker \delta_1$. Thus there exists $b \in B^{\#}$ such that $\langle b \rangle$ centralizes $H/H \cap \ker \delta_1$. Therefore $[H, \langle b \rangle] \leqslant H \cap \ker \delta_1$. But H is A-invariant and $\langle b \rangle \lhd A$; so $[H, \langle b \rangle] = [H, \langle b \rangle]^{a_i} \leqslant (H \cap \ker \delta_1)^{a_i}$, $1 \leqslant i \leqslant e$. Thus $[H, \langle b \rangle] \leqslant \bigcap_{i=1}^{e} (H \cap \ker \delta_1)^{a_i} = 1$; i.e., $\langle b \rangle$ centralizes H. Thus $A/\langle b \rangle$ acts fixed-point-freely on H. If $\langle b \rangle = B$, $A/\langle b \rangle \simeq Q$; so H is nilpotent by Thompson's theorem. If $\langle b \rangle < B$, $A/\langle b \rangle$ is a Frobenius group with cyclic kernel and complement of order q, and $| A/\langle b \rangle | < \mathscr{N}$ since $| A | \leqslant \mathscr{N}$. Thus $f(H) \leqslant l(A/\langle b \rangle)$ by definition of $\mathscr{N}$; so $f(H) \leqslant l(A) - 1$.

If (c) holds, $A_1 = Q$ and the only divisor of $| Q |$ other than 1 is q. If p is a prime such that $p^f + 1 = q$ for some positive integer f, then $q \neq 2$, so q is odd and $p^f = q - 1$ is even. Thus $p = 2$ and $q = 2^f + 1$ is a Fermat prime. Thus if $2 \nmid | H/H \cap \ker \delta_1 |$, by Lemma I.3.3, Q does not act faithfully on $H/H \cap \ker \delta_1$. In fact, so long as $H/H \cap \ker \delta_1$ has no extra-special section of order 2^{2f+1} if $q = 2^{f+1}$, Q does not act faithfully on $H/H \cap \ker \delta_1$ [12, pp. 716–717]. Thus by hypothesis (1), Q centralizes $H/H \cap \ker \delta_1$. This means that $[H, Q] \leqslant H \cap \ker \delta_1$. Now $B \lhd A$; so $C_H(B)$ is A-invariant. In particular, Q acts on $C_H(B)$, and Q acts fixed-point-freely on $C_H(B)$ since $A = BQ$ acts fixed-point-freely on H. Thus $[C_H(B), Q] = C_H(B)$ [7, 5.3.5], and $C_H(B) = [C_H(B), Q] \leqslant [H, Q] \leqslant H \cap \ker \delta_1$. Thus $C_H(B) = C_H(B)^{a_i} \leqslant (H \cap \ker \delta_1)^{a_i}$, $1 \leqslant i \leqslant e$, and $C_H(B) \leqslant \bigcap_{i=1}^{e} (H \cap \ker \delta_1)^{a_i} = 1$; i.e., B acts fixed-point-freely on H. Then by Berger's result [2, p. 305], $f(G) \leqslant l(B) = l(A) - 1$.

If (d) holds, A_1 is a Frobenius group with cyclic kernel of order $| B_1 | < | B |$ and complement of order q. Thus any divisor of $| B_1 |$ is a proper divisor of $| B |$. Then since we have hypotheses (2) and (3), by Theorem I.3.5 applied to $H/H \cap \ker \delta_1$, A_1, V_1, K in place of H, A, V, K, A_1 is not faithful on $H/H \cap \ker \delta_1$. Then by Lemma I.3.4(2) there exists a $b \in B^{\#}$ such that $\langle b \rangle$ centralizes $H/H \cap \ker \delta_1$. Thus $f(H) \leqslant l(A) - 1$, as shown in case (b) above, and Theorems 1.6 and 1.4 are established.

2. Fixed-Point-Free Action of a Group of Order pq

LEMMA 2.1. *Let G be a finite solvable group admitting a cyclic group Q of prime order, q, such that $q \nmid |G|$. Suppose W is a finite-dimensional KGQ-module and W_G is a faithful KG-module, where* char $K = r \nmid |G|_q$ *and K is a splitting field for all the subgroups of GQ. Then if Q acts fixed-point-freely on V and G possesses no extra-special section of order 2^{2f+1} such that $q = 2^f + 1$, a Fermat prime, then $[G, Q] = 1$.*

Proof. By Maschke's theorem, $W = W_1 \oplus \cdots \oplus W_j$, a direct sum of irreducible KGQ-modules. Let α_i be the representation of GQ on W_i; so $\bigcap_{i=1}^{j} \ker \alpha_i = 1$ since GQ is faithful on W by Lemma I.3.2. Now $(G/G \cap \ker \alpha_i)Q$ acts faithfully on W_i by Lemma I.3.2; so W_i is a faithful, irreducible $K(G/G \cap \ker \alpha_i)Q$-module, and Q acts fixed-point-freely on W_i. Then since $G/G \cap \ker \alpha_i$ possesses no extra-special section of order 2^{2f+1} such that $q = 2^f + 1$, $[G/G \cap \ker \alpha_i, Q] = 1$ [12, pp. 716–717]. Thus $[G, Q] \leqslant G \cap \ker \alpha_i$, $1 \leqslant i \leqslant j$; so $[G, Q] \leqslant \bigcap_{i=1}^{j} (G \cap \ker \alpha_i) = 1$; i.e., $[G, Q] = 1$.

LEMMA 2.2. *Suppose $G = STQ$ where*

(1) *$S \lhd G$ and S is an extra-special 2-group;*

(2) *T is an elementary Abelian t-group, $t \neq 2$, such that $[S, T] = S$ and $[T, Q] = T$;*

(3) *$|Q| = q$, a prime distinct from 2 and t.*

Suppose further that V is a finite-dimensional KG-module, where K is a splitting field for all subgroups of G and char $K = r$ *is distinct from 2 and q. Then Q does not act fixed-point-freely on V if G acts faithfully on V.*

Proof. Suppose the lemma is false; so Q acts fixed-point-freely on V. Since T is Abelian and $[T, Q] = T$, $C_T(Q) = 1$; so TQ is a Frobenius group. Thus if W is an irreducible LTQ-module on which T is nontrivial, where L is a splitting field for TQ such that char $L \nmid |TQ|$, then by Lemma I.2.1, $\dim_L W = q$ [4, 13.8]. Now $[S/Z(S), T] = S/Z(S)$; so T acts fixed-point-freely on $S/Z(S)$, an $\mathbb{F}_2 TQ$-module. Since $2 \nmid |TQ|$, $S/Z(S)$ is a completely reducible $\mathbb{F}_2 TQ$-module, and T acts fixed-point freely on each irreducible $\mathbb{F}_2 TQ$-submodule of $S/Z(S)$. Thus q divides the dimension over $\mathbb{F}_2$ of each irreducible $\mathbb{F}_2 TQ$-submodule of $S/Z(S)$ [4, 24. 14]. Let $|S| = 2^{2n+1}$. Then $q \mid 2n = \dim_{\mathbb{F}_2}(S/Z(S))$. Since $q \neq 2$, $q \mid n$. Thus $n = mq$ for some integer m, and $|S| = 2^{2mq+1}$. Now the proofs of Lemmas I.2.5 and I.2.6 show that $|C_{S/Z(S)}(Q)| = |S/Z(S)|^{1/q}$ since T is nontrivial on $S/Z(S)$, T is Abelian, and TQ is a Frobenius group. Thus $|C_{S/Z(S)}(Q)| = 2^{2m}$. Now $[S, Q] \lhd S$; so $Z(S) \leqslant [S, Q]$, and $[S/Z(S), Q] = [S, Q]/Z(S)$. Now $S/Z(S) = [S/Z(S), Q] \times C_{S/Z(S)}(Q)$; so $|[S/Z(S), Q]| =$

$|S/Z(S)|/|C_{S/Z(S)}(Q)| = 2^{2m(q-1)}$, and $|[S,Q]| = 2^{2m(q-1)+1}$. Now $[S,Q]$ is extra-special [7, p. 215, No. 11]; so $Z(S)$ is its unique minimal normal subgroup. $[S,Q]Q$ acts faithfully on V, which is a completely reducible $K[S,Q]Q$-module since $r \neq 2$ or q. Thus there exists some irreducible $K[S,Q]Q$-submodule U of V on which $Z(S)$ is nontrivial; so $[S,Q]$ is faithful on U. Thus $[S,Q]Q$ is faithful on U by Lemma I.3.2, and Q acts fixed-point-freely on U. Then necessarily $|[S,Q]| = 2^{2f+1}$, where $q = 2^f + 1$ [12, pp. 716–717]. Thus $2m(q-1)+1 = 2f+1$; so $f = m(q-1)$ and $q = 2^{m(q-1)} + 1$, where $m \geqslant 1$. But clearly $2^{q-1} > q$ for $q \geqslant 3$; this contradiction establishes the lemma.

THEOREM 2.3. *Let $A = PQ$ be a Frobenius group with kernel P of prime order, p, and complement Q of prime order q. Suppose G is a finite solvable group such that $(|G|, |A|) = 1$ and A acts fixed-point-freely on G. Then $f(G) \leqslant l(A) = 2$.*

Proof. It is Shult's proof of this theorem, with the added hypothesis that $2 \nmid |G|$ if q is a Fermat prime, upon which the proof of Theorem 1.4 is based. We now analyze the proof of Theorem 1.4 in the special case that $B = P$ of prime order and see that the prime condition is unnecessary.

The reduction from Theorem 1.4 to Theorem 1.6 did not involve any prime conditions; these came from the later application of the representation theorems of Part I. Thus to prove Theorem 2.3 it is sufficient to prove the following:

THEOREM 2.4. *Let $A = PQ$ be a Frobenius group with kernel P of prime order, p, and complement Q of prime order, q. Let H be a finite solvable group such that $(|H|, |A|) = 1$ and A acts fixed-point-freely on H. Suppose F is a finite-dimensional, faithful KHA-module, where K is a finite field of characteristic r such that $r \nmid |A|$ and $O_r(H) = 1$. If A acts fixed-point-freely on F, then $f(H) \leqslant l(A) - 1 = 1$; i.e., H is nilpotent.*

Proof. Let H, A, F, K be a counterexample to Theorem 2.4 with $|H|$ as small as possible. Then if $H_0 < H$, $H_0 \lhd H$ and H_0 is A-invariant, $O_r(H_0) = 1$; so H_0, A, F, and K satisfy the hypotheses of the theorem and H_0 is nilpotent by minimality of $|H|$. Suppose M is a proper normal A-invariant subgroup of H maximal with respect to that property. Then H/M is solvable and characteristically simple and therefore an elementary Abelian t-group for some prime t. Since $|M| < |H|$, M is nilpotent. Clearly $M = F(H)$. Since $O_r(M) = 1$, M is an r'-group. Let T be the unique A-invariant Sylow t-subgroup of H [7, 6.2.2(i), (ii)]; so $H = MT$.

Suppose $H_0 < H$ and H_0 is A-invariant. If $r \nmid |H|$, then $O_r(H_0) = 1$; so H_0 is nilpotent since $H_0 < H$. If $r \mid |H|$, since M is an r'-group and H/M is a t-group, $t = r$ and $M = O_{r'}(H)$. Then $T \simeq H/M$ and T is irreducible under A

since H/M is. Then since $O_r(H_0)$ is an A-invariant r-subgroup of H, $O_r(H_0) \leqslant T$ [7; 6.2.2(iii)]; so $O_r(H_0) = 1$ or $O_r(H_0) = T$. If $O_r(H_0) = 1$, H_0 is nilpotent as above. If $O_r(H_0) = T$, since $M = O_{r'}(H)$ is a Hall r'-subgroup of H, $O_{r'}(H_0) = H_0 \cap M$ is a Hall r'-subgroup of H_0, and $H_0 = (H_0 \cap M)T$. Now $[H_0 \cap M, T] = [O_{r'}(H_0), O_r(H_0)] = 1$; so $H_0 = (H_0 \cap M) \times T$ and is therefore nilpotent. Thus whether or not $r \mid |H|$, if $H_0 < H$ and H_0 is A-invariant, H_0 is nilpotent.

Since $H = MT$ is not nilpotent and $O_s(H)$ is a Sylow s-subgroup of H for all $s \neq t$, there exists an s such that T does not centralize $S = O_s(H)$. But S char H; so ST is A-invariant and not nilpotent. Therefore, $H = ST$. Now if $S_0 < S$ and S_0 is TA-invariant, S_0T is a proper subgroup of H and S_0T is A-invariant; so S_0T is nilpotent; i.e., $[S_0, T] = 1$. Since H is not nilpotent, T does not centralize S, but does centralize any TA-invariant proper subgroup S_0 of S. Since $(|S|, |TA|) = 1$, S is a special s-group [7, 5.3.7]. Also, if $[S, T] < S$, since $[S, T]$ is TA-invariant, $[[S, T], T] = 1$. But then $[S, T] = 1$ [7, 5.3.6], a contradiction. Thus $[S, T] = S$. If $T_0 < T$, $T_0 \lhd T$, and T_0 is A-invariant, ST_0 is A-invariant and proper in H; so ST_0 is nilpotent. Thus T_0 char $ST_0 \lhd ST = H$; so $T_0 \leqslant O_t(H)$. Therefore, $T/O_t(H)$ is irreducible under A. Now $O_t(H) \leqslant C_T(S) < T$, and $C_T(S)$ is A-invariant; so $O_t(H) = C_T(S)$; i.e., $T/O_t(H)$ acts faithfully on S. Thus $[S, T/O_t(H)] = [S, T] = S$. Since $P \lhd A$ and $T/O_t(H)$ is irreducible under A, $C_{T/O_t(H)}(P) = 1$ or $T/O_t(H)$.

Now fix $H = ST$ and consider a counterexample H, A, F, K to Theorem 2.4 with $\dim_K F$ as small as possible. The same arguments used in the proof of Theorem 1.6 show that we may assume K is a splitting field for all the subgroups of HA, and that F is an irreducible KHA-module. The same analysis of the e Wedderburn components of F with respect to H yields cases (a), (b), and (c); case (d) is impossible since P contains no nontrivial proper subgroup. Cases (a) and (b) are eliminated without prime restriction just as in the proof of Theorem 1.6.

We use the notation of the proof of Theorem 1.6. If case (c) holds, necessarily $2 \mid |H|$ and $q = 2^f + 1$, a Fermat prime, while $H/H \cap \ker \delta_1$ possesses an extra-special section of order 2^{2f+1}. Since $A_1 = Q$, $e = |A : A_1| = pq/q = p$. Q fixes V_1 and $A = PQ$ permutes the V_i, $1 \leqslant i \leqslant p$, transitively; so P permutes them transitively. Thus for each i, $1 \leqslant i \leqslant p$, there exists a unique element b_i of P such that $V_1^{b_i} = V_i$. Now if Q centralizes $H/H \cap \ker \delta_1$, we are done as in the proof of Theorem 1.6. Thus Q acts faithfully on $H/H \cap \ker \delta_1$. Also, since H is faithful on $F = V_1 \oplus \cdots \oplus V_p$ and $(H \cap \ker \delta_1)^b$ is the kernel of the action of H on $V_1{}^b$, $\bigcap_{b \in P} (H \cap \ker \delta_1)^b = 1$.

If $H_0 \leqslant H$, let $\bar{H}_0 = H_0(H \cap \ker \delta_1)/(H \cap \ker \delta_1)$. Thus $\bar{H}Q$ acts faithfully as well as irreducibly on V_1 by Lemma I.3.2. Suppose $H_0 \leqslant H$, H_0 is A-invariant, and $\bar{H}_0$ is an Abelian subgroup of $\bar{H}$ such that $r \nmid |\bar{H}_0|$. Now $\bar{H}_0Q$ acts faithfully on V_1 since $\bar{H}Q$ does; so by Lemma 2.1, $[\bar{H}_0, Q] = 1$. Thus $[H_0, Q] \leqslant H_0 \cap \ker \delta_1$. But $C_{H_0}(P) = [C_{H_0}(P), Q]$ since $A = PQ$ acts fixed-point-freely on H_0;

so $C_{H_0}(P) \leqslant H_0 \cap \ker \delta_1$. Thus $C_{H_0}(P) = \bigcap_{b \in P} (C_{H_0}(P))^b \leqslant \bigcap_{b \in P} (H_0 \cap \ker \delta_1)^b = 1$; i.e., $C_{H_0}(P) = 1$. Thus since $\overline{Z(S)}$ is Abelian and Q-invariant and $r \neq s$, $C_{Z(S)}(P) = 1$. Then $Z(S) \cap C_S(P) = 1$; so $C_S(P) \cong C_S(P)Z(S)/Z(S) \leqslant S/Z(S)$, which is elementary Abelian since S is special. Therefore, $C_S(P)$ is Abelian, and $C_S(P)$ is A-invariant since $P \lhd A$; so $\overline{C_S(P)}$ is Abelian and Q-invariant. Then since $r \neq s$, $C_{C_S(P)}(P) = 1$; so $C_S(P) = 1$.

Now if $C_{T/O_t(H)}(P) = 1$, $C_{S(T/O_t(H))}(P) = 1$, and by Thompson's theorem $S(T/O_t(H))$ is nilpotent. This implies $[S, T/O_t(H)] = 1$, contradicting the fact that $[S, T/O_t(H)] = S \neq 1$. Thus $C_{T/O_t(H)}(P) = T/O_t(H)$, and $C_T(P)\, O_t(H) = T$. Since $O_t(H)$ centralizes S, $C_T(P)$ does not centralize S; so $SC_T(P)$ is an A-invariant subgroup of H that is not nilpotent. Thus $SC_T(P) = H = ST$ by minimality of H, and $C_T(P) = T$. Since $T > 1$, $Z(T) > 1$; so if $t \neq r = \operatorname{char} K$, $[Z(T), Q] \leqslant H \cap \ker \delta_1$, and $C_{Z(T)}(P) = 1$ as shown above. But $C_T(P) = T$; so $C_{Z(T)}(P) = Z(T) \neq 1$, a contradiction. Thus $t = r$ and T is elementary Abelian.

Now if $s \neq 2$ or $\bar{S}$ is Abelian, $[\bar{S}, Q] = 1$ by Lemma 2.1. Since Q centralizes $\bar{S}$, $[\bar{T}, Q]$ centralizes $\bar{S}$. But $T = C_T(P)$ implies $[T, Q] = T$; so $[\bar{T}, Q] = \bar{T}$, and $\bar{T}$ centralizes $\bar{S}$. Since $[S, T] = S$, $[\bar{S}, \bar{T}] = \bar{S}$; so $\bar{S} = 1$; i.e., $S \leqslant H \cap \ker \delta_1$. But then $S = \bigcap_{b \in P} S^b \leqslant \bigcap_{b \in P} (H \cap \ker \delta_1)^b = 1$, a contradiction. Therefore, $s = 2$ and $\bar{S}$ is non-Abelian.

Now $S \lhd HQ$ and V_1 is an irreducible KHQ-module. Thus if W is a Wedderburn component of V_1 with respect to S, Q stabilizes W. (If not, take $w \neq 0$ in W. Then $\sum_{y \in Q} w^y \neq 0$ and is a fixed point for Q.) Thus if $x \in T$, Q stabilizes the Wedderburn component W^x, and for $y \in Q$, $W^{[x,y]} = (W^{x^{-1}})^{y^{-1}xy} = W^y = W$; i.e., $[T, Q] = T$ stabilizes W. Thus $STQ = HQ$ stabilizes W; i.e., V_1 is a homogeneous KS-module. Therefore, V_1 is a faithful, homogeneous $K\bar{S}$-module; so there exists a faithful, irreducible representation of $\bar{S}$ over K, and $Z(\bar{S})$ is cyclic. Now $\bar{S}$ is a non-Abelian 2-group such that $Z(\bar{S})$ is cyclic, and $\bar{S}/Z(\bar{S})$ is elementary Abelian since $S/Z(S)$ is elementary Abelian. Then since $\bar{S} = [\bar{S}, \bar{T}]$, $\bar{S}$ is an extra-special 2-group [7, p. 215, No. 11]. But then by Lemma 2.2, Q cannot act fixed-point-freely on V_1. This contradiction eliminates case (c), and the theorem is proved.

Note. If PQ acts fixed-point-freely on G, then Q acts fixed-point-freely on $C_G(P)$, which is therefore nilpotent by Thompson's theorem. Kurzweil has proved that if $f(C_G(P)) \leqslant 1$, $f(G) \leqslant 3$. Here our stronger hypothesis leads us to $f(G) \leqslant 2$. See *Math Z.* 120 (1971) for Kurzweil's results.

COROLLARY 2.5. *Let A be a group with $l(A) \leqslant 2$, and G be a finite solvable group such that $(|G|, |A|) = 1$ and A acts fixed-point-freely on G. Then $f(G) \leqslant l(A)$.*

Proof. If $l(A) = 1$, Thompson's theorem applies. If $l(A) = 2$ and A is Abelian, Berger's result applies. If A is non-Abelian, Theorem 2.4 applies.

3. Fixed-Point-Free Action of Groups of Order *upq*

It is easy to establish the following lemma and its corollary.

LEMMA 3.1. *Let X be a group of order upq, where u, p, and q are three distinct primes. If X is non-Abelian, it is of one of the following three types:*

(1) *the direct product of a group of prime order and a Frobenius group with kernel and complement each of prime order;*

(2) *a Frobenius group with kernel of prime order and cyclic complement;*

(3) *a Frobenius group with cyclic kernel and complement of prime order.*

COROLLARY 3.2. *Let X be a group of order upq, where u, p, and q are distinct primes. If Y is a subgroup of prime order in X, either $Y \lhd X$ or Y has a normal complement in X.*

We are now in a position to show that, with certain restrictions on the primes dividing $|G|$, if X acts fixed-point-freely on a solvable group G, $(|G|, |X|) = 1$, and $|X| = upq$, the product of three distinct primes, then $f(G) \leqslant l(X) = 3$. In proving Theorem 1.4, we used the fact that A was a Frobenius group with cyclic kernel and complement of prime order in three different ways:

(1) In the reduction to Theorem 1.6 and the argument showing F is an irreducible KHA-module, where K is a splitting field for all the subgroups of A, we used only the fact that if $1 \leqslant N \lhd A$, then A/N is a Frobenius group of the same type as A but with smaller order, or A/N is of prime order. Then we used Thompson's theorem and induction on $|A|$ to show that if A/N acts fixed-point-freely on a solvable group G_0 of order prime to $|A/N|$, then $f(G_0) \leqslant l(A/N)$.

(2) In the analysis of the possibilities for A_1, we eliminated $A_1 = A$ by Theorem I.1.2 and used the fact that any proper subgroup of A is either cyclic or a Frobenius group of the same type as A. Then we used the representation theorems from Part I—this was the only place where we needed the hypotheses concerning prime divisors of $|G|$.

(3) After application of the representation theorems, we used the fact that for any subgroup J of A, if $1 \neq M \lhd J$, either M contains a nontrivial normal subgroup N of A or there exists a cyclic normal complement D to M in A. The former possibility occurred in cases (b) and (d) and led to $C_H(A/N) = 1$ and $f(H) \leqslant l(A) - 1$ as in (1) above. The latter occurred in case (c) and led to $C_H(D) = 1$ and $f(H) \leqslant l(D) \leqslant l(A) - 1$ by Berger's result.

Now suppose X is a group of order upq, u, p, and q distinct primes. Then a theorem like Theorem 1.4 can be proved since X has the following properties:

(1′) If $1 < N \lhd X$, then $l(X/N) \leqslant 2$; so if X/N acts fixed-point-freely on a solvable group G_0 of order prime to $|X/N|$, then $f(G_0) \leqslant l(X/N)$ by Corollary 2.5.

(2′) Any proper subgroup Y of X is such that $l(Y) \leqslant 2$, and Y is cyclic or a Frobenius group with kernel and complement each of prime order. Thus after property (1′) has been used to reduce the analog of Theorem 1.4 to an analog of Theorem 1.6, we can apply Lemma I.3.3 and Theorem I.3.5 [12, pp. 716–717] as in the proof of Theorem 1.6—it is only here that we need prime restrictions, and these are completely determined by the isomorphism types of the proper subgroups of X, to which the representation theorems are applied.

(3′) Suppose $J \leqslant X$ and $1 \neq M \lhd J$. Then either $M \lhd X$ or M has a normal complement D in X, for either $M = X \lhd X$, or M or its complement is of prime order and Corollary 3.2 applies. Clearly if $C_H(A/M) = 1$ or $C_H(D) = 1, f(H) \leqslant l(A) - 1$ by Corollary 2.5. Thus the method of proof of Theorems 1.4 and 1.6 yields the following theorem:

THEOREM 3.3. *Suppose X is a group of order upq, where u, p, and q are distinct primes, and G is a finite solvable group such that $(|G|, |X|) = 1$ and X acts fixed-point-freely on G. Suppose further that:*

(1) *If Y is a cyclic proper subgroup of X, $d \mid |Y|$, and $d = s^f + 1$ for some prime s, and positive integer f, then G possesses no extra-special section of order s^{2f+1}.*

(2) *If Y is a proper subgroup of X and Y is a Frobenius group with kernel of order t and complement of order r ($t, r \in \{u, p, q\}$), then*

(a) *If $t = (s^n + 1)/(s^{n/r} + 1)$ for some prime s and integer n, then G posesses no extra-special section of order s^{2n+1}.*

(b) *If $t = (s^m - 1)/(s^{m/r} - 1)$ for some prime s and integer m, then G possesses no elementary abelian section of order s^{2m}.*

Then $f(G) \leqslant l(X) = 3$.

We conclude with an example that illustrates a method for using Theorem 3.3 and is pleasantly easy to state.

EXAMPLE 3.4. Let X be a group of order 30 and let G be a (solvable) group such that $(|G|, 30) = 1$ and X acts fixed-point-freely on G. Then $f(G) \leqslant l(X) = 3$.

Proof. If Y is a cyclic proper subgroup of X, $|Y| = 2, 3, 5, 6, 10$, or 15. There exists no prime s and positive integer f such that $s^f + 1 = 2$ or 15. If $s^f + 1 = 3$ or 5, $s = 2$; so $s \nmid |G|$. If $s^f + 1 = 10$, $s = 3$; so $s \nmid |G|$. If $s^f + 1 = 6$, $s = 5$; so $s \nmid |G|$. Thus hypothesis (1) of Theorem 3.3 is satisfied.

If Y is a proper subgroup of X that is a Frobenius group, it is dihedral group; so Theorem I.3.5 holds for Y without prime restrictions; i.e., hypotheses (2)(a) and (b) of Theorem 3.3 are satisfied, and we are done.

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References

1. T. R. Berger, Class two p groups as fixed point free automorphism groups, *Illinois J. Math.* **14** (1970), 121–149.
2. T. R. Berger, Nilpotent fixed point free automorphism groups of solvable groups, *Math. Z.* **131** (1973), 305–312.
3. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1966.
4. L. Dornhoff, "Group Representation Theory, Part A," Dekker, New York, 1971.
5. W. Feit, "Characters of Finite Groups," Benjamin, New York, 1967.
6. G. Glauberman, Correspondences of characters for relatively prime operator groups, *Canad. J. Math.* **20** (1968), 1465–1488.
7. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.
8. R. Griess, Automorphisms of extra special groups and nonvanishing degree 2 cohomology, *Pacific J. Math.* **48** (1973), 403–422.
9. N. Jacobson, "Lectures in Abstract Algebra II," Van Nostrand, Princeton, N.J., 1953.
10. I. Kaplansky, "Fields and Rings," Univ. of Chicago Press, Chicago, 1969.
11. E. Shult, Nilpotence of the commutator subgroup in groups admitting fixed point free operator groups, *Pacific J. Math.* **17** (1966), 323–347.
12. E. Shult, On groups admitting fixed point free Abelian operator groups, *Illinois J. Math.* **9** (1965), 701–720.
13. J. Thompson, Finite groups with fixed point free automorphisms of prime order, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 578–581.

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